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Course material

Lessons and Exercises in Financial Mathematics

Third-Semester Courses for Second-Year Bachelor's Students

In Financial Sciences and Accounting

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Preface

This course material has been developed as a comprehensive module in *Financial Mathematics* at Oran 2 Mohamed Ben Ahmed University. It comprises **seven chapters** covering both theoretical and practical aspects of the subject. The material designed to provide students with a structured, accessible, and pedagogically sound learning resource that facilitates the acquisition of fundamental concepts and their practical applications in financial mathematics. The work represents the culmination of several years of continuous effort and draws upon extensive teaching experience acquired through lectures, tutorial sessions, and practical exercises. This experience has enabled the development of an approach that conveys financial concepts with clarity, precision, and academic rigor.

The module is organized around two principal themes: **short-term financial operations** and **long-term financial operations**. Each chapter combines theoretical explanations with numerical examples, analytical demonstrations of key formulas, and exercises aimed at reinforcing understanding and developing problem-solving skills.

A distinctive feature of this course material is that it is written in **English**, supported by an extensive range of pedagogical and professional references in banking and finance. This approach is intended to provide students, fellow instructors, and professionals with access to international financial terminology and to foster comparative perspectives between the **Anglo-Saxon** and **Francophone** approaches to financial education and practice.

The preparation of this module has benefited greatly from the constructive feedback of colleagues and students, whose contributions have significantly enhanced the clarity, relevance, and pedagogical value of the material presented.

Houda Djaballah

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Introduction

Financial mathematics occupies a central position in the fields of economics and management sciences, as it provides the essential analytical tools for understanding and evaluating the mechanisms that govern financial operations. In an increasingly complex and globalized economic environment, mastering the principles of financial calculation has become indispensable for students of economics, management, commerce, and accounting. Indeed, sound decision-making in areas such as investment, financing, and banking requires a solid understanding of the time value of money and the quantitative methods used to measure it.

This course material, entitled “*Lessons and Exercises in Financial Mathematics*”, is the result of several years of teaching experience in this subject, combined with an extensive review of numerous specialized references. It is distinguished by the fact that it has been entirely prepared in English - the language of international banking and finance - enabling students to strengthen both their technical and linguistic competencies.

The content has been developed according to a pedagogical approach that emphasizes clarity, gradual progression, and scientific rigor. The theoretical concepts are presented in a straightforward manner, supported by numerous applied examples and well-structured exercises designed to facilitate comprehension and mastery of financial problem-solving techniques.

To achieve the intended learning objectives, the material is organized around two main axes:

- **Short-term financial operations**, which include simple interest, discounting under simple interest, and capital equivalence under simple interest.
- **Long-term financial operations**, which include compound interest, discounting under compound interest, capital equivalence under compound interest, annuities, and loan amortization (indivisible loans).
- **Investment project selection (capital budgeting)**, which uses discounted cash flow techniques to evaluate and compare investment alternatives and

support optimal financial decision-making. This work aims to serve not only as an effective learning aid but also as a practical reference for both students and professionals seeking to develop their analytical skills and deepen their understanding of financial mechanisms within the context of a globalized economy.

Chapter 01

Simple Interest

Simple Interest

Borrowing money involves a cost. The price paid for the use of money is called **interest**. The amount borrowed or lent is referred to as the **principal** or **present value**.

Simple interest is calculated only on the original principal. When money is borrowed, the borrower typically pays a fixed interest rate on the principal for the entire period during which the funds are used. Although the interest rate is usually quoted on an annual basis, it may also be expressed for shorter periods such as a week, a month, or a quarter.

Key Point :

Borrowing money has a cost called **interest**, and under **simple interest** it is calculated only on the original **principal** at a fixed rate over time.

Simple Interest Formula

The formula for simple interest helps you find the interest amount if the principal amount **p**, **r** rate of interest and **t** time period are given.

Simple interest formula (in years) is given as:

$$I = \frac{Pxrxt}{100}$$

$$P = \frac{100 \times I}{rxt}$$

$$r = \frac{100 \times I}{Pxt}$$

$$t = \frac{100 \times I}{Pxr}$$

Where:

I = Simple Interest

P = Principal, Principal is denoted by **P** or **C** (Capital)

r = Interest rate (in percentage), Rate is the rate of interest at which the principal amount is given to someone for a certain time, is denoted by **r**

t = time duration (**in years**), Time is the duration for which the principal amount is given to someone. Time is denoted by **t** or **n**.

Example:

Mourad's father had borrowed personal loan of **1.000** DZD from the bank and the rate of interest was **5%**. What would the simple interest be if the amount is borrowed for **1 year**?

Solution:

$$I = 1.000 \times 5 \times 1 / 100 = 50 \text{ DZD}$$

Note: If the duration is in days, the duration is expressed as a fraction of a year. The base being the number of conventional days in a year, it is written as:

The most common cases are:

Base = 360: referred to as the commercial base (banking year);

Base = 365 or 366: referred to as the exact base (calendar year).

Depending on the unit used to characterize t (number of periods), this general formula $I = P \times t \times r / 100$ becomes:

$$\text{In months: } I = P \times r \times \text{number of months} / 1.200$$

$$\text{In days: } I = P \times r \times \text{number of days} / 36.000.$$

360 days is commonly used in many financial calculations.

Example: Let's calculate the interest owed by a company that received a loan of 250.000 DZD from **June 20 to August 15** of the same year at a rate of 11% (360-day year).

Solution:

The number of placement days is determined day by day, so:

$$t = (30-20) + 31 + 15 = \mathbf{56 \text{ days.}}$$

$$I = P \times r \times \text{number of days} / 36.000.$$

$$I = 250.000 \times 11 \times 56 / 36.000 = \mathbf{4.277,78 \text{ DZD.}}$$

Note:

A year is classified as a **leap year** if it is **divisible by four (4)** without leaving a remainder, meaning the result of the division is a **whole number**. This rule implies that every fourth year typically includes an **extra day** in February, making the total number of days in the year **366** instead of **365**.

However, an **exception** applies to **century years** (years ending in “00”). Such years are **not considered leap years** unless they are also **divisible by 400**.

Example:

$2024 \div 4 = 506$ (no remainder) $\rightarrow$ 2024 is a **leap year**.

Example:

$1900 \div 4 = 475$.

but $1900 \div 400 = 4.75$ (not an integer) $\rightarrow$ 1900 is a **common year**.

Example :

$$2000 \div 4 = 500$$

$2000 \div 400 = 5$ (no remainder) $\rightarrow$ 2000 is a **leap year**.

The **future value (FV)** is the total amount of money that will be accumulated after interest is added to the principal, also called the **accumulated value**. The formula to calculate future value is:

$$FV = P + I$$

Where:

- I : Simple Interest
- P : Principal

So, the formula becomes:

This can be simplified to:

$$FV = P \times (1 + rt)$$

Example:

If you invest 1.000 DZD at a simple interest rate of 5% (0.05) for 10 years:

1. Calculate interest : $1.000 \times 5 \times 10 / 100 = 500$ DZD
2. Future value : $1.000 + 500 = 1.500$ DZD

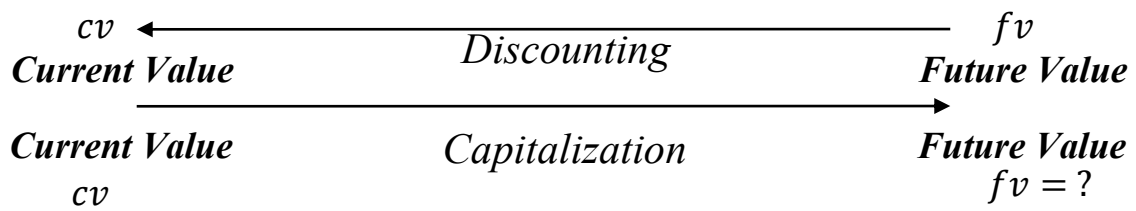
So, the future value would be **1.500 DZD** after 10 years.

Concepts: Discounting and Capitalization

According to the above, the interest rate appears as the rate of transformation of money over time. This **relationship** between **time** and **interest rate** means that two sums of money are only equivalent if they are equal on the same date. Therefore, in order to compare two or more amounts available at different dates, the use of actuarial calculation techniques (capitalization and discounting) becomes necessary.

Capitalization refers to the process of determining the future value of a current sum of money based on a specific interest rate over time. It essentially "grows" the investment by applying interest.

Discounting, on the other hand, involves calculating the present value of future cash flows, taking into account a specific discount rate. This process "reduces" future amounts to reflect their worth today.



Tutorial Sheet n° 01: Simple Interest

Exercise1:

An individual invests 45.000 MU for three (03) months, starting from June 10. The interest rate is 10% per year. How much does he have at the end of the investment?

Exercise2:

An individual invests 75.000 MU from May 15 to September 18, in an account yielding 9.5% per year. How much does he have at the end of the investment?

Exercise 3:

An individual deposits 27.000 MU in an account. One hundred (100) days later, he withdraws 28.140 MU. What is the interest rate?

Exercise 4:

An individual invests 29.200 MU in an account yielding 11.40%. At the end of the investment, he finds 31.428,45 MU. What was the duration of the investment?

Exercise 5:

A principal of 7.200 MU, lent at an interest rate of 8% on June 8, has acquired a value of 7.288 MU at the end of the loan. Determine when the loan was repaid?

Exercise6:

Calculate the principal that, invested at 8.4% for 62 days, has acquired a value of 16.738,70 MU.

Solution

Exercise 1:

To find the Future Value of the investment, we can use the following two formulas:

$$FV = P + I$$

$$I = P \times r \times \text{number of months} / 1200$$

$$I = \frac{p.r.t}{1.200} =$$

Now we can substitute the values into the formula to find I

$$\frac{45.000 \times 10 \times 3}{1.200} = \mathbf{1.125}$$

Therefore, we find the following result: $FV = 45.000 + 1.125 = \mathbf{46.125 \text{ MU}}$.

At the end of this investment, he will have **46.125 MU**.

Exercise 2:

Time duration:

$$t = 16 + 30 + 31 + 31 + 18 = \mathbf{126 \text{ jours}}$$

$$I = P \times r \times \text{number of days} / 36000$$

$$I = \frac{75.000 \times 9,5 \times 126}{36.000} = 2.493,75 \text{ MU}$$

$$FV = 75.000 + 2.493,75 = 77.493,75 \text{ MU}$$

At the end of this investment, he will have **77.493,75 MU**

Exercise 3 :

$$I = \frac{pxrxt}{36000}$$

$$I = 28.140 - 27.000$$

$$I = 1.140 \text{ MU}$$

$$I = \frac{27.000 \times r \times 100}{36.000} = 27000$$

$$2.700.000 r = 41.040.000$$

the interest rate is : $r = 15,2 \%$

Exercise 4 :

To find the duration of the investment, we can use the following two formulas:

$$I = \frac{pxrxt}{36000}$$

$$FV = P + I$$

Where:

- FV = interest rate (11.40% or 0.114)
- t = time in years ?
- I = simple interest

$$FV = P + I \quad \text{based on that:} \quad I = FV - P = \frac{pxrxt}{36000}$$

$$31.428,45 - 29.200 = 29.200 \times 0,114 \times \frac{n}{360}$$

$$t = \frac{31.428,45 - 29.200}{29.200} \times \frac{1}{0,114} \times 360 = 241 \text{ days}$$

The duration of the investment is approximately **0.67 years**, which is about **8 months**

Exercise 5 :

To resolve the exercise, we need to determine the duration of the loan based on the given information:

- **Principal (P):** 7,200 MU
- **Interest Rate (r):** 8% or 0.08
- **Future value :** 7,288 MU

We can use the simple interest formula:

$$FV = P + I$$

Based on that:

$$P = FV - I$$

$$I = 7.288 - 7.200 = 88 \text{ MU}$$

$$\text{Duration of loan : } t = \frac{36.000 \cdot I}{P \cdot r} = \frac{36.000 \times 88}{7.200 \times 8} = 55 \text{ days}$$

The duration of the loan is 55 days

Loan Repayment Date:

Since the loan was taken out on **June 8**, we can add 55 days to that date to find the repayment date:

June 8 + 55 days = August 02.

Thus, the loan was repaid on August 02.

Exercise 6 :

To find the duration of the investment, we can use the following two formulas:

$$I = \frac{pxrxt}{36000}$$

$$FV = P + I$$

So : $FV = P + \frac{pxrxt}{36000}$

Where:

FV = the total amount (future value) = 16,738.70 MU

P = principal (the amount we want to find) ?

r = interest rate (8.4% or 0.084)

t = time in days

Now we can substitute the values into the formula to find **P**:

$$P + \frac{P \times 8,4 \times 62}{36.000} = 16.738,70$$

.....By solving the equation, we find

$$P = 16.500 \text{ U/M}$$

The principal that, invested at an interest rate of 8.4% for 62 days, has acquired a value of **16,738.70 MU** is approximately **16,500 MU** .

Average rate of Interest and Effective rate

The Average Investment Rate

When **Capitals** are invested at different **Rates**, an **average rate** for these investments is defined. **The average investment rate** is the single rate at which the capitals would need to be invested to achieve the same total interest, while keeping the durations of the investments unchanged.

To find the **Average Investment Rate**, we can use the following formula:

$$R = \frac{\sum_{i=1}^n P_i r_i t_i}{\sum_{i=1}^n P_i t_i}$$

Where:

R : is the average interest rate.

P_i : is the capital amount for each investment.

r_i : is the interest rate for each investment.

t_i : is the duration for each investment.

The Proof:

Based on the fact that the Average Interest Rate achieves the same amount of interest that could be gained from investing the same amounts at different rates, we write:

$$I = I_1 + I_2 + I_3 + \dots + I_n$$

$$\frac{P_1 R t_1}{36000} + \frac{P_2 R t_2}{36000} + \dots + \frac{P_n R t_n}{36000} = \frac{P_1 r_1 t_1}{36000} + \frac{P_2 r_2 t_2}{36000} + \dots + \frac{P_n r_n t_n}{36000}$$

$$R(P_1 t_1 + P_2 t_2 + \dots + P_n t_n) = P_1 r_1 t_1 + P_2 r_2 t_2 + \dots + P_n r_n t_n$$

$$R = \frac{P_1 r_1 t_1 + P_2 r_2 t_2 + \dots + P_n r_n t_n}{P_1 t_1 + P_2 t_2 + \dots + P_n t_n}$$

$$R = \frac{\sum_{i=1}^n P_i r_i t_i}{\sum_{i=1}^n P_i t_i}$$

Example:

Let's consider the following investments:

1. **First Investment:** Principal amount of 18,500 DZD at an interest rate of 5% from January 17, 2019 to May 24, 2019.
2. **Second Investment:** Principal amount of 11,500 DZD at an interest rate of 3,5 % from March 20, 2019 to June 28, 2019.
3. **Third investment:** Principal amount of 28,400 DZD at an interest rate of 7% from April 1 to August 20, 2019.

Calculate the average investments rate.

Solution:

For Investment A, the duration is: **t1 = 127 days**

For Investment B, the duration is: **t2 = 100 days**

For Investment C, the duration is: **t3 = 141 days**

$$R = \frac{\sum_{i=1}^n P_i r_i t_i}{\sum_{i=1}^n P_i t_i}$$

$$R = \frac{18500 \times 5 \times 127 + 11500 \times 3,5 \times 100 + 28400 \times 7 \times 141}{18500 \times 127 + 11500 \times 100 + 28400 \times 141}$$

$$R = 5,83\%$$

Types of interest rate:

Interest rates can be classified into several types depending on how they are calculated and applied. The **simple interest rate** is based only on the original principal, while the **compound interest rate** considers both the principal and the accumulated interest, leading to higher returns over time. The **nominal interest rate** represents the stated annual rate without considering compounding, whereas the **effective interest rate** reflects the true annual cost or yield after accounting for compounding periods. The **discount rate** is used to determine the present value of a future amount, particularly in short-term financial operations. In contrast, the **real interest rate** adjusts the nominal rate for inflation, showing the actual increase in purchasing power. Finally, some loans or investments use a **floating (variable) interest rate**, which changes according to market or

policy rate fluctuations. Together, these different types of rates help measure and compare the cost of borrowing and the return on investment in various financial contexts.

Effective Interest Rate

In the context of simple interest, interest can either be paid **in advance** (at the time of the loan disbursement) or **upon repayment** of the loan. These two methods are not financially equivalent. By convention, the **effective simple interest rate** refers to the simple interest rate with interest paid at the time of loan repayment.

The **effective interest rate** (viewed as post-calculated interest) of a **pre-calculated interest** operation is higher than the announced interest rate. The formula to convert a nominal rate to an effective rate is as follows:

$$\text{EIR} = r / (1 - t \times r / 100)$$

Where:

EIR: effective simple interest rate

t: time in year

r: nominal interest rate

Explanation:

This formula allows you to convert the nominal rate of a pre-calculated operation into an effective rate, which is higher because the loan amount is reduced by the interest that is deducted upfront. Therefore, when moving from a pre-calculated rate to an effective (post-calculated) rate, the effective rate will be higher.

Example:

Suppose you take out a loan of **10.000 DZD** with a nominal annual interest rate of **10%**. However, this interest is **paid in advance** (pre-calculated), meaning the lender deducts the interest upfront from the loan amount. The loan term is **1 year**.

Step 1: Calculate the upfront interest

With a 10% nominal interest rate on a 10.000 DZD loan, the interest amount deducted upfront is:

$$\text{Interest} = 10 \times 10.000 \times 1/100 = \mathbf{1.000 \text{ DZD}}$$

So, instead of receiving 10.000 DZD you receive:

$$\text{Amount received} = 10.000 - 1000 = 9.000 \text{ DZD}$$

Step 2: Calculate the effective interest rate

To find the effective interest rate, use the formula:

$$\mathbf{EIR = r / (1 - t \times r / 100)}$$

$$\text{EIR} = 10 / (1 - 1 \times 10 / 100)$$

$$\text{EIR} = 10 / 0.9$$

$$\mathbf{\text{EIR} = 11.11\%}$$

So, the effective interest rate is **11.11%**, which is higher than the nominal rate of **10%**, because the interest was deducted in advance.

This shows how the effective interest rate increases when the interest is paid upfront.

Note: Discounted interest or prepaid interest: This refers to the interest that is deducted from the principal at the beginning of the investment period.

Example:

Let's assume someone invests 1.000 DZD at a **nominal interest rate** of 5% for **one year**, but the interest of 50 is deducted upfront, leaving an actual investment of 950 DZD.

- **Nominal interest rate:** 5%
- **Nominal amount:** 1.000 DZD
- **Prepaid interest:** 50 DZD (5% of 1.000 DZD)
- **Actual invested amount:** 950 DZD

At the end of the year, the total interest earned is still based on the nominal amount, which is 50 DZD. But since the actual investment was only 950 DZD, the **effective interest rate (EIR)** is:

$$\text{EIR} = r / (1 - t \times r / 100)$$

$$\text{EIR} = 5 / (1 - 1 \times 5 / 100)$$

$$= 5.26 \%$$

So, even though the **nominal interest rate** is 5%, the **effective interest rate** (or **true interest rate**) is 5.26% due to the fact that the investment was reduced by prepaid interest.

Prepaid interest or **discounted interest** is used in many Anglo-Saxon countries, especially in the United States and the United Kingdom. This type of interest is common in several financial contexts, particularly in **short-term loans**, **discount bonds**, and **Treasury bills**.

This system helps lenders or investors to receive interest upfront and is a common mechanism for managing liquidity and ensuring guaranteed returns.

Tutorial Sheet n° 02: The Average Investment Rate and the Effective Interest Rate

Exercise 1

Calculate the average rate resulting from the following investments:

No.	Principal	Rate	Period
1	3,800 MU	7.5%	May 25 to July 15
2	6,420 MU	8.2%	May 25 to July 31
3	780 MU	8.5%	May 25 to August 31

Exercise2

An amount of 20 000 MU is invested at an interest rate paid in advance of 9% for 20 months.

Calculate the effective interest rate resulting from the operation.

Exercise3

A loan of 300 000 MU is granted at a rate of r %. After 4 months, the borrower repays 120 000 monetary units of principal to his lender, which the lender immediately reinvests at 9%.

After one year (from the initial operation), the lender receives the full amount of principal and interest and notes that his principal has ultimately been invested at an average rate of $(r - 0.8)$ %.

1- Calculate r .

2- What is the total amount the lender has at the end of the year?

Exercise4

Two amounts, with a total value of 16 800 MU, are invested for one year at respective rates that differ by 0.40%. The total interest = 1651, 20 MU. If the first amount had been invested at the second rate, and the second amount at the first rate, the annual total interest would have been 1641, 60 MU.

Calculate the two amounts and the two rates.

Solutions**Exercise 1**

Investment duration:

No. 1 = 51 days; No. 2 = 67 days; No. 3 = 98 days.

Average investment rate R :

$$R = \frac{\sum_{i=1}^n P_i r_i t_i}{\sum_{i=1}^n P_i t_i}$$

$$(3800 \times 7.5 \times 51) + (6420 \times 8.2 \times 67) + (780 \times 8.5 \times 98) / (3,800 \times 51) + (6420 \times 67) + (780 \times 98) = 8.04\%$$

$$R = 8.04\%$$

Exercise 2

There are two methods to solve this exercise: one is direct, and the other is indirect

Indirect method (3 steps)

- Interest generated by the investment (prepaid interest)

$$I = (20\,000 \times 9 \times 20) / 1\,200 = 3\,000 \text{ MU}$$

$$\text{Prepaid interest} \quad 3\,000 \text{ MU}$$

- Effectively invested principal P_0

$$P_0 = P - I$$

$$20\,000 - 3\,000 = 17\,000 \text{ MU}$$

Effective invested principal 17 000 MU

$$P_0 = 17\,000 \text{ MU}$$

- Effective investment rate **EIR**

$$\frac{P_0 \times \text{EIR} \times t}{1200} = I \quad \text{therefore}$$

$$\frac{17\,000 \times \text{EIR} \times 20}{1200} = 3\,000$$

$$\text{EIR} = 1\,200 \times 3\,000 / 17\,000 \times 20 = 10.59\%$$

$$\text{EIR} = 10.59 \%$$

Direct method

To find the effective interest rate, use the formula:

$$\text{EIR} = r / (1 - (t \times r / 100))$$

Numerical application

$$EIR = 9 / (1 - (20 \times 9 / 1200))$$

$$EIR = 10.59\%$$

Exercise 3

Determining the rate r

Period 1: For the first 4 months, the lender has 300 000 MU invested at rate $r\%$.

Period 2: After 4 months, the borrower repays 120 000 MU, leaving 180 000 MU still invested at rate $r\%$, while the 120 000 MU is reinvested by the lender at 9% for the remaining 8 months.

300 000 MU at rate r for **4 months**

180 000 MU at rate r for **8 months**

120 000 MU at rate **9%** for **8 months**

this produces the same interest as the investment of:

300 000 MU at the rate **($t - 0.8$)** for **12 months**.

We can write:

$$\begin{aligned} \frac{300.000 \times r \times 4}{1.200} + \frac{180.000 \times r \times 8}{1.200} + \frac{120.000 \times 9 \times 8}{1.200} \\ = \frac{300.000 \times (r - 0,8) \times 12}{1.200} \end{aligned}$$

Thus, $r = 12\%$.

Determining the total amount FV

Acquired value= **Future Value** = principal + Interest

$$300\,000 + (300\,000 \times (12 - 0.8) \times 1) / 100 = 333,600 \text{ MU}$$

Exercise 4

Let P and $P_2 = (16\,800 - P)$ be the invested amounts, r and $r_2 = (r - 0.4)$ be the corresponding rates.

We can write:

$$\begin{aligned} \frac{P \cdot r}{100} + \frac{(16.800 - P)(r - 0,4)}{100} &= 1.651,20 \text{ or } 16.800 r + 0,4P \\ &= 171.840 \end{aligned}$$

and

$$\frac{P(r - 0,4)}{100} + \frac{16.800 - P}{100} = 1.641.60 \text{ or } 16.800r - 0,4P = 164.160$$

Now we have a system of two equations:

$$1. \quad 16\,800r + 0.4P = 171\,840$$

$$2. 16\,800r - 0.4P = 164\,160$$

We can add these two equations to eliminate P

$$(16\,800r + 0.4P) + (16\,800r - 0.4P) = 171\,840 + 164\,160$$

This simplifies to:

$$33\,600r = 336\,000$$

$$r = 336\,000 / 33\,600 = 10\%$$

$$r = 10\%$$

We have $r = 10\%$, we can substitute it back into one of the original equations to find P.

$$P = 9600 \text{ MU}$$

We can find P_2 , r_2

$$P_2 = 16\,800 - P = 16\,800 - 9\,600 = 7200 \text{ MU}$$

$$P_2 = 7200 \text{ MU}$$

$$R_2 = 10\% - 0.4\% = 9.6\%$$

Results

$P = 9\,600 \text{ MU}$ (the first amount)

$P_2 = 7\,200 \text{ MU}$ (the second amount)

$r = 10\%$ (the interest rate for the first amount)

$r_2 = 10\% - 0.4\% = 9.6\%$ (the interest rate for the second amount)

Thus, 9,600 U/M was invested at 10%, and 7,200 U/M was invested at 9.6%.

Chapter 02

Simple Discounting

Simple Discounting

In business transactions, companies frequently buy and sell goods and services on credit. To formalize and secure these credit arrangements, negotiable instruments known as commercial instruments are used. The most common commercial instruments are the bill of exchange and the promissory note.

Types of commercial instruments

The term "**bill of exchange**" refers to a negotiable commercial instrument that evidences a debt and facilitates commercial credit transactions. It is a written order by which a creditor, known as the **drawer**, instructs a debtor, known as the **drawee**, to pay a specified sum of money on a predetermined future date (the maturity date) to a designated beneficiary, known as the **payee**, or to the holder of the instrument.

The commercial instrument can be transferred by simple endorsement, either by the bank itself or by the bank's client if the client retains ownership.

Key points

A bill of exchange is a legal instrument whereby a debtor undertakes to pay a specified amount of money at a future date (the maturity date).

A **promissory note** is a **written promise** by one party (the note's **maker**) to pay another party (the note's **payee**) a definite **sum of money**, either on demand or at a **specified future date**. A promissory note typically contains all the terms involved, such as the principal debt amount, interest rate, maturity date, payment schedule, the date and place of issuance, and the issuer's signature.

Key points

A promissory note is a written and unconditional promise by one party to pay a specified amount of money to another party at a predetermined date or upon demand

Difference Between a Promissory Note and a Bill of Exchange

Promissory Note	Bill of Exchange
A promise to pay	An order to pay
Two parties (maker and payee)	Three parties (drawer, drawee, and payee)
Signed by the debtor	Issued by the creditor
The maker undertakes to pay	The drawer orders the drawee to pay

Discount and Proceeds

Companies often require short-term liquidity. Banks may intervene by purchasing negotiable financial instruments before their maturity date in exchange for a discount charge (agio). This operation is called discounting.

Key points

Discounting concerns short-term financial instruments payable at a future date, which can be converted into cash before maturity through a bank, at a discount.

How to Calculate Simple Discount?

In financial mathematics, **simple discount** refers to the process where the interest (discount) is **subtracted** from the principal amount before the actual transaction takes place. It's often used in cases such as **treasury bills** or when someone is **receiving a payment before it's due**.

Key points

In the discounting of commercial paper, the bank purchases the instrument before its maturity date by paying its present value, which is obtained by deducting the discount (interest in advance). At maturity, the bank receives the full face value from the debtor, and the discount represents the bank's profit.

Formula for Simple Discount

The formula to calculate the **discount amount** is:

$$D = FV \times d \times t$$

Where:

1. ***D*** : is the total discount (bank discount)
1. ***FV*** : is the maturity value, which is the amount that will be paid at the end of the term.
2. ***d***: is the discount rate.
3. ***t*** : is the term of the discount : **time** (usually in years or fraction of a year)

Key points

In financial mathematics, time is commonly expressed using a 360-day year or a 12-month year. When time is measured in days, it is written as $t=d/360$, and when measured in months, it is written as $t=m/12$

Calculating the Proceeds (Actual Amount Received):

The proceeds (C or P), also called the present value, are obtained by subtracting the discount (D) from the future value (FV), so that :

$$P = FV - D$$

Where:

P = Proceeds (present value / cash received now)

FV = Maturity value

D = Discount

Example 1: Megan would like to borrow 1500 DZD from her local bank for 1,5 years at a discount rate of 9,75 %. How much would Megan receive as proceeds? How much would the bank earn as a discount?

Solution:

$$1- \text{ the discount : } D = FV \times d \times t$$

$$D = 1500 \times 1.5 \times 9,75/100 = 219,73 \text{ DZD}$$

$$2- \text{ the Proceeds : } P = FV - D = 1500 - 219,73 = 1280,6 \text{ DZD}$$

Example2:

Let's say someone issues a note with a maturity value (face value) of 10000 DZD, with a discount rate of 5% per year, and the note is due in 6 months (0.5 years).

1. Calculate the Discount Amount

$$D = FV \times d \times t \quad 10000 \times 5 \times 0.5 / 100 = 250 \text{ DZD}$$

So, the discount amount is **250 DZD**

2. Calculate the Proceeds

$$P = FV - D \quad 10000 - 250 = 9750 \text{ DZD}$$

So, the borrower receives **9750 DZD** .

Example 3:

Mouhamed borrowed 14,750 DZD from Walid at an interest rate of 7%, repayable in 90 days, and signed a promissory note to that effect. However, 30 days later, Walid needed cash urgently and could not wait until the maturity date for repayment. He therefore presented the promissory note to his local bank and discounted it at a discount rate of 8.25%. How much cash did the bank provide to Walid, and how much did the bank earn from this transaction?

Solution:

- Calculate the maturity value of the promissory note

The maturity value (FV) is the amount Mohamed will repay at the end of 90 days. The face value of the promissory note is 14,750 DZD. We first compute the interest using simple interest.

$$\text{Interest} = \text{Face Value} \times \text{Interest Rate} \times \text{Time}$$

$$\text{Interest} = 14,750 \text{ DZD} \times 0.07 \times 90 / 360 = 258,125 \text{ DZD}$$

$$\text{Maturity Value (FV)} = \text{Face Value} + \text{Interest} = 14,750 \text{ DZD} + 258,125 \text{ DZD} = 15,008,13 \text{ DZD}$$

- Calculate the discount amount

Next, we calculate the discount amount deducted by the bank when Walid discounts the note after 30 days.

$$\text{Discount Amount (D)} = \text{Maturity Value} \times \text{Discount Rate} \times \text{Time until maturity}$$

$$D = 15,008,125 \times 0.0825 \times (60/360) \approx 206,36 \text{ DZD}$$

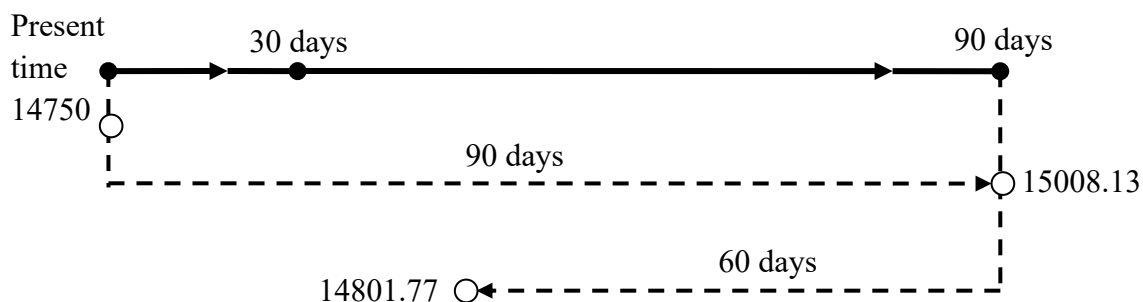
- Calculate the cash received by Walid

$$\text{Cash} = \text{Maturity Value} - \text{Discount Amount}$$

$$\text{Cash} = 15,008,13 \text{ DZD} - 206,36 \text{ DZD} \approx 14,801,77 \text{ DZD}$$

Cash received by Walid: approximately 14,801,32 DZD

Bank's profit from the transaction: approximately 206,36 DZD



Tutorial Sheet n° 03: Simple Discounting

Exercisen°01

On August 22, a commercial instrument with a maturity date of November 30 and a maturity value of 12,000 MU is discounted at a commercial discount rate of 9%.

Calculate the commercial discount and the proceeds (the present commercial value of this instrument).

Exercisen°02

A bill of exchange with a maturity value of 40,000 MU and a term of 4 months is negotiated. Its discount is equal to 771.10 MU.

1. Calculate the discount rate.
2. Calculate the proceeds.
3. Calculate the rational discount.

Exercisen°03

A discount request was submitted on March 31, including three commercial bills, each with a nominal value of 6,600 UM. The total discount is calculated at a rate of 8.5% and amounted to 280.50 UM.

Determine the maturity date of the third bill, knowing that the first one is due on April 30 and that the discount for the second bill amounted to 93.50 MU.

Exercisen°04

A bill maturing on June 30 was discounted at a 9.2% rate on May 19. Another bill, with the same maturity date, was discounted at a 9.5% rate on June 2. If the discount rates for the two bills are swapped, the total proceeds (present values) remain unchanged.

Calculate the respective maturity values (nominal value) of the two bills, knowing that their total is 85,000 MU.

Solutions**Exercise n°01**

- Maturity value $FV=12\ 000\ MU$
- Annual discount rate $d=9\%$
- Due date = maturity date November 30
- Discount date: August 22.

Using the commercial year of 360 days, the time between August 22 and November 30 is **100 days**. $t=100\text{days}$.

Calculate the commercial discount

$$D = \frac{FV \times d \times t}{36.000}$$

$$\frac{12.000 \times 9 \times 100}{36.000} = \mathbf{300\ MU}$$

Calculate the proceeds (or present value)

Note: *Proceeds* are often denoted by the letter **P**, **PV** (Present Value) or **C** for cash proceeds.

The proceeds **P** are the difference between the maturity value and the discount amount:

$$P = FV - D$$

Substitute the values:

$$P = 12\ 000 - 300 = \mathbf{11\ 700\ MU}$$

Exercise n° 02

Calculate the Discount Rate

The formula for the commercial discount, calculated on a **per-month** basis, is as follows:

$$D = \frac{FV \times d \times t}{1200}$$

Where:

- $D=771.10\ MU$
- $FV=40\ 000\ MU$

Substitute the known values:

$$771,10 = \frac{40.000 \times r \times 4}{1.200}$$

$$\approx 0.05783$$

So, the discount rate is approximately **5.78%**.

Calculate the Proceeds

The proceeds P are calculated using the formula:

$$P = FV - D$$

Substitute the known values:

$$P = 40\,000 - 771,10 = 39\,228,90$$

So, the proceeds are **39,228.90** monetary units.

$$40.000 - 771,10 = \mathbf{39.228,90 \text{ U/M}}$$

3 – Calculate the rational discount

It can be found in **two ways**, but **the first is obligatory** and the **second is optional**, noting that the law of the first method was proven in the lecture. ¹

First method: Using the formula directly

$$D_r = \frac{FV \times d \times t}{36.000 + d \times t}$$

$$D_r = \frac{40000 \times 5,78 \times 4}{1200 + 5,78 \times 4}$$

$$\mathbf{576,099..... \approx 576,10 \text{ MU}}$$

Second method: Indirect

- First, we must find the present value, and based on that, the rational discount is calculated.

$$PV + \frac{PV \times 5,78 \times 4}{1.200} = 40.000$$

$$\frac{1.200PV}{1.200} + \frac{PV \times 5,78 \times 4}{1.200} = 40.000$$

$$\frac{1.200PV + 23,12PV}{1.200} = 40.000$$

$$1.223,12 \text{ PV} = 40.000 \times 1.200$$

$$\text{PV} = \frac{48.000.000}{1.223,12}$$

$$\text{PV} = 39.243,90 \text{ U/M}$$

- **Rational discount**

$$D_r = \frac{\text{PV} \cdot d \cdot t}{1.200}$$

$$D_r = \frac{39243,90 \times 5,78 \times 4}{1.200}$$

$$D_r = 756,10 \text{ U/M}$$

Exercise n° 3

Calculate the discount for the first bill

For the first bill, the discount formula is:

$$D1 = \frac{6.600 \times 8,5 \times 30}{36.000} = 46,75 \text{ MU}$$

The discount for the third bill **D3** is:

$$D3 = 280.50 - (46.75 + 93.50) = 140.25$$

$$D3 = 140.25 \text{ MU}$$

Maturity date of the third bill:

$$t = \frac{36.000 D}{FV \cdot d}$$

$$t = \frac{36.000 (140,25)}{6.600 \times 8,5} = 90 \text{ days}$$

Determine the maturity date of the third bill

The discount submission was made on March 31. Adding 90 days to this date gives us the due date for the third bill:

March 31 + 90 days = June 29

The maturity date of the **third bill** is **June 29**.

Exercise n° 4**Calculate the Time to Maturity**

For both bills, the maturity date is June 30:

- For Bill 1, discounted on May 19, the time to maturity is the difference between May 19 and June 30, which is **42 days**
- For Bill 2, discounted on June 2, the time to maturity is the difference between June 2 and June 30, which is **28 days**

Let:

- $FV1 = x$ be the Maturity value of Bill 1,
- $FV2 = y$ be the maturity value of Bill 2,
- The total maturity value: $F1+F2=85,000$ MU

If the **total of the present values** of the two bills **remains unchanged**, it means that **the total of the discounts applied has not changed**.

$$\frac{x \times 9,2 \times 42}{36000} + \frac{y \times 9,5 \times 28}{36000} = \frac{x \times 9,5 \times 42}{36000} + \frac{y \times 9,2 \times 28}{36000}$$

$$x \times 42 \times (9,5 - 9,2) = y \times 28 \times (9,5 - 9,2)$$

$$(1) \quad 2y = 3x$$

The combination of this equation (1) and the equation:

$$(2) \quad x+y=85\,000$$

Leads to :

$$x = 34.000 \text{ MU} \quad \text{et} \quad y = 51.000 \text{ MU}$$

maturity value of Bill 1: 34 000 monetary units

maturity value of Bill 2: 51 000 monetary units

Rational Discount and Banking Fees

Rational Discount

Commercial discount is often considered less realistic because it is based on nominal (face) values, which do not reflect the true value of the financial instrument at the discounting date. In contrast, rational discount provides a more accurate approach, as it is based on the present value of the instrument at the time of the transaction.

The rational discount is generally denoted by D_r or D' .

The rational discount formula

$$D_r = \frac{FV \times d \times t}{36.000 + d \times t}$$

Where:

$D_r = D'$: Rational discount

d : discount rate

t : The duration between the discount date and the maturity date

FV : The maturity value of the commercial instrument

Key points

Commercial discount is widely used in **short-term banking operations** such as bills of exchange and treasury bills, although it is considered a traditional and less precise method. In contrast, **rational discount** is mainly used in **financial theory** and **valuation models**, as it is based on present value principles and provides a more accurate economic assessment.

Example:

Company X discounted a bill of exchange with a Maturity value of 100000 DZD at a discount rate of 9% per year, maturing in 66 days.

1. Calculate the commercial discount D and the present commercial value (proceeds) P .
2. Calculate the rational discount D_r and the present rational value.
3. Compare the two discounts.
- 4.

Solution:

1. Calculate the commercial discount and the present commercial value:

$$D = \frac{FV \times d \times t}{36.000}$$

$$D = \frac{100000 \times 9 \times 66}{36.000} = 1650 \text{ DZD}$$

2. Calculate the rational discount and the present rational value:

$$D' = D_r = \frac{FV \times d \times t}{36.000 + d \times t}$$

$$D' = D_r = \frac{100000 \times 9 \times 66}{36.000 + 9 \times 66}$$

$$D' = D_r = \frac{59\,400\,000}{36\,594} = 1623,22 \text{ DZD}$$

3. Comparison of the Discounts:

Commercial Discount: 1650 DZD; Proceeds (present value): 98 350 DZD.

Rational Discount: 1623.22 DZD; rational present Value: 98 376, 78 DZD.

We notice that the **commercial discount** is greater than the **rational discount**, which is in favor of the bank rather than the holder of the commercial instrument.

The proof:

The rational present value of the commercial instrument is denoted by **PV**:

$$D_r = \frac{PV \times d \times t}{36.000}$$

$$PV = FV - D_r$$

Therefore:

$$FV = PV + D_r = PV + \frac{PV \times d \times t}{36.000}$$

Thus, extracting a common factor (PV) and then unifying the denominators, we find:

$$FV = PV \left(1 + \frac{d \times t}{36.000}\right)$$

$$FV = PV \left(\frac{36.000 + dt}{36.000} \right)$$

$$PV = \frac{36.000 \times FV}{36.000 + dt}$$

By dividing the numerator and denominator by **d**, the formula can be expressed in another way:

$$PV = \frac{\frac{36.000 \times FV}{d}}{\frac{36.000 + dt}{d}}$$

$$PV = \frac{Dev \times FV}{Dev + t}$$

Where:

$$Devisor = 36000 / d$$

We have:

$$D_r = FV - PV$$

$$D_r = FV - \frac{36.000 \times FV}{36.000 + dt}$$

$$D_r = \frac{36.000 \times FV + FV \times dt - 36.000 \times FV}{36.000 + dt} = \frac{FV \times dt}{36.000 + dt}$$

$$D_r = \frac{FV \times d \times t}{36.000 + dt}$$

By dividing the numerator and denominator by **d**, the formula can be expressed in another way:

$$D_r = \frac{(FV \times dt)/d}{(36.000 + dt)/d}$$

$$D_r = \frac{FV \times t}{Dev + t}$$

Banking fees (bank charges or Agios)

In English-speaking countries, terms like "**bank charges**", "**banking fees**", are more commonly used instead of "agios"

When the bank discounts commercial bills (commercial instruments), it does not only deduct the discount amount from the maturity value of the commercial instrument. In addition to the proper commercial discount, the bank also deducts the following items:

- ✓ Various commissions;
- ✓ Value Added -Tax (VAT).
- **Duration-dependent commissions** are calculated using the same principle as the discount formula. They are therefore proportional both to the time interval between the discount date and the maturity date of the commercial instrument, and to its maturity value. These commissions are denoted by C_d .
- There are also **duration-independent commissions**, which are proportional only to the maturity value of the commercial instrument and are denoted by C_i .
- In addition, **fixed fees** may be applied. These fees are independent of both the maturity value and the duration of the instrument.
- Regarding **Value Added Tax (VAT)**, the currently applicable rate in Algeria is 19%. However, interest, discount charges, as well as endorsement and acceptance fees are exempt from VAT. Consequently, the taxable base consists of duration-independent commissions and fixed fees subject to taxation. The corresponding VAT is collected by the bank on behalf of the tax authorities and is applied to the total amount of these taxable charges. *The Agios is calculated using the following formula:*

Banking Fees (Agios)= Discount (D)+ Duration-dependent Commission (C_d)+ duration-Independent Commission(C_i) + Value- Added Tax (VAT)

$$\text{Agios} = D + C_d + C_i + VAT$$

$$\text{Agios} = \frac{FVdt}{36000} + \frac{FVd't}{36000} + \frac{FVk}{100} + vat$$

Thus, the Proceeds P is calculated as follows:

$$P = FV - \text{Agios}$$

Key points

Agios represent the total banking charges and are computed as the sum of the discount, duration-dependent commissions, duration-independent commissions, and applicable Value Added Tax (VAT).

Effective Discount Rate (EDR)

The **Effective Discount Rate (EDR)** represents the actual rate of discount applied to a financial instrument, such as a bill of exchange, after incorporating additional banking costs such as commissions and other charges. It reflects the true cost of discounting the instrument.

The EDR formula is as follows:

$$EDR = \frac{36000 \times \text{agios}}{fv \times t}$$

Note:

Value Added Tax (VAT) is not included in the calculation of the effective discount rate because it is generally recoverable. Although VAT is initially deducted during the discounting operation, it is subsequently refunded to the taxpayer through the VAT declaration process.

Proof of the formula

Given :

D: Discount

Ca: Duration-Dependent Commission

C_i: Duration- Independent Commission

d: Discount Rate

d': Duration-dependent Commission Rate

k: Duration- Independent Commission Rate

t: duration of discount

We start from the expression of Agios:

$$\text{Agios} = \frac{FVdt}{36000} + \frac{FVd't}{36000} + \frac{FVk}{100}$$

By **multiplying** by **360 t** and dividing by **360 t**, the equation becomes as follows:

$$A_{gios} = \frac{FVdt}{36000} + \frac{FVd't}{36000} + \frac{360FVkt}{36000t}$$

Factorizing :

$$A_{gios} = \frac{FV \cdot t}{36000} * \left[d + d' + \frac{360k}{t} \right]$$

$\left[d + d' + \frac{360k}{t} \right]$ This part represents *EDR*: the effective Discount Rate.

Accordingly:

$$A_{gios} = \frac{FV \cdot t \cdot (EDR)}{36000}$$

Therefore:

$$EDR = \frac{36000 \times agios}{fv \times t}$$

Example

On July 4, a bill of exchange with a face value (maturity value) of **6,000 DZD** is discounted at a bank under the following conditions:

- Discount rate: **10.5%**
- Duration-dependent commission rate: **0.6%**
- Duration-independent commission rate: **1/8%**
- Value Added Tax (VAT): **19%**

The maturity date of the bill is **November 12 of the same year**.

Solution

Face Value (V) maturity value= 6000 DZD

Discount Rate (d) = 10,5% = 0,105

Duration-Dependent Commission Rate (d') = 0,6% = 0,006

Duration-Independent Commission Rate (k) = 1/8% = 0,00125

Value Added Tax (VAT) = 19% = 0,19

Maturity Date = November 11

Discount Date = July 4

Calculation of the Time to Maturity

The time from July 4 to November 11 is:

July: 27 days (from July 4 to July 31)

August: 31 days

September: 30 days

October: 31 days

November: 12 days

Total days = 27 + 31 + 30 + 31 + 12 = **131 days**

Calculation of the Discount Amount

The discount amount (***D***) can be calculated using the formula:

$$D = Fv \times d \times t$$

$$D = 6000 \times 0,105 \times 131/360$$

$$D = 6000 \times 0,105 \times 0,4444 \approx \mathbf{229.5 \text{ DZD}}$$

Calculation of the Duration-Dependent Commission

We calculate the duration-dependent commission (***C_d***) accurately:

$$C_d = FV \times d' \times 160/360$$

$$C_d = 6000 \times 0,006 \times 131/360$$

$$C_d = 6000 \times 0,006 \times 0,4444 \approx \mathbf{13.1 \text{ DZD}}$$

Calculation of the Duration-Independent Commission

The duration-independent commission (***C_i***) can be calculated as:

$$C_i = FV \times k$$

$$6000 \times 0,00125 = \mathbf{7,5 \text{ DZD}}$$

Calculation of the VAT

The VAT can be calculated as follows:

$$VAT = (C_i) \times VAT \text{ rate}$$

$$7,5 \times 0,19 \approx \mathbf{1,425 \text{ DZD}}$$

Calculate the Total Agios

Now we can calculate the total Agios:

$$Agios = D + C_d + C_i + VAT$$

$$= 229.5 + 13.1 + 7,5 + 1,425 \approx \mathbf{251, 525 \text{ DZD}}$$

Calculation of the Proceeds (P)

$$P = FV - \text{Agios}$$

$$6000 - 254,014 \approx 5748,475 \text{ DZD}$$

Calculation of the Effective Discount Rate (EDR)

The real discount rate can be calculated as:

$$EDR = \frac{36000 \times \text{agios}}{fv \times t}$$

Note : Agios Excluding VAT

$$EDR = 36000 \times 250,1 / 6000 \times 131 \approx 11.45\%$$

To confirm we use the formula

$$EDR = \left[d + d' + \frac{360k}{t} \right] = \left[10.5 + 0.6 + \frac{360 \times 1/8}{131} \right] \approx 11.45 \%$$

Chapter 03:

Capital Equivalence

under Simple Interest

Capital Equivalence under Simple Interest

Definition of the Principle of Equivalence

The Principle of Equivalence states that two or more financial obligations are considered equivalent if they have the same value at a specified focal date, determined using the same interest or discount rate.

Importance and Practical Applications of the Principle of Equivalence

The Principle of Equivalence is used to replace a debt, a commercial paper, or a set of payments with another financial obligation without disadvantaging either party. It ensures that the financial value remains unchanged at a specified focal date, thereby facilitating fair financial adjustments and simplifying the management of debts and receivables.

Practical applications include:

- Replacing one commercial paper with another having a different maturity date.
- Consolidating several debts or commercial papers into a single obligation.
- Rescheduling debts when the debtor faces financial difficulties.
- Converting a single payment into several payments, or vice versa.
- Modifying loan terms, such as changing the number or timing of installments.

Equivalence of Commercial Instruments

The equivalence of commercial instruments refers to the process of determining a common reference date at which two or more financial instruments, differing in nominal value and maturity, have the same present value when discounted at a given interest rate.

Key Points

Present Value: It involves calculating the present value of each instrument to assess their values at the same time.

Time Value of Money: It emphasizes that money today is worth more than the same amount in the future, requiring consideration of the time until maturity.

Comparison of Different Instruments: This concept applies to various commercial instruments like promissory notes and bills of exchange.

Common Date: Equivalence is established at a shared date for straightforward comparison.

Key rule to remember:

All commercial instruments are compared at a single reference date, and all amounts must be evaluated at that date using the same interest or discount rate.

Equivalence formula

Let us consider the following data for two commercial instruments:

1. First Commercial Instrument

Maturity value: FV_1

Time to maturity: t_1

Discount: D_1

Present value (Proceeds): PV_1

2. Second Commercial Instrument

○ Maturity value: FV_2

○ Time to maturity: t_2

○ Discount: D_2

○ Present value (Proceeds): PV_2

The divisor, denoted by D_{EV} , is given by $D_{EV} = 36000/d$ where d represents the discount rate.

We say that the first instrument is equivalent to the second if both are discounted at the same discount rate on a specific reference date. The condition for equivalence is as follows:

$$\frac{FV_1}{FV_2} = \frac{(D_{EV} - t_2)}{(D_{EV} - t_1)}$$

Proof

Since, at the equivalence date, the present values of the two instruments are equal, we can write the following:

$$PV_1 = PV_2$$

we also have $D = \frac{FV \times d \times t}{36.000}$

By dividing the numerator and denominator by d , the formula becomes:

$$D = \frac{FV \cdot t}{D_{EV}}$$

Note: Devisor is denoted by D_{EV} which is $36000 / d$

Therefore:

$$D_1 = \frac{FV_1 \cdot t_1}{D_{EV}} \quad \text{and} \quad D_2 = \frac{FV_2 \cdot t_2}{D_{EV}}$$

$$FV_1 - D_1 = FV_2 - D_2$$

$$FV_1 - \frac{FV_1 \cdot t_1}{D_{EV}} = FV_2 - \frac{FV_2 \cdot t_2}{D_{EV}}$$

$$\frac{FV_1 \cdot D_{EV} - FV_1 \cdot t_1}{D_{EV}} = \frac{FV_2 \cdot D_{EV} - FV_2 \cdot t_2}{D_{EV}}$$

$$FV_1(D_{EV} - t_1) = FV_2(D_{EV} - t_2)$$

$$\frac{FV_1}{FV_2} = \frac{(D_{EV} - t_2)}{(D_{EV} - t_1)}$$

Example: Two commercial instruments with the following maturity values and maturity dates are considered:

First instrument:

$FV_1 = 1586,40$ MU

Maturity Date: 07/31

Second instrument:

$FV_2 = 1592,40$ MU

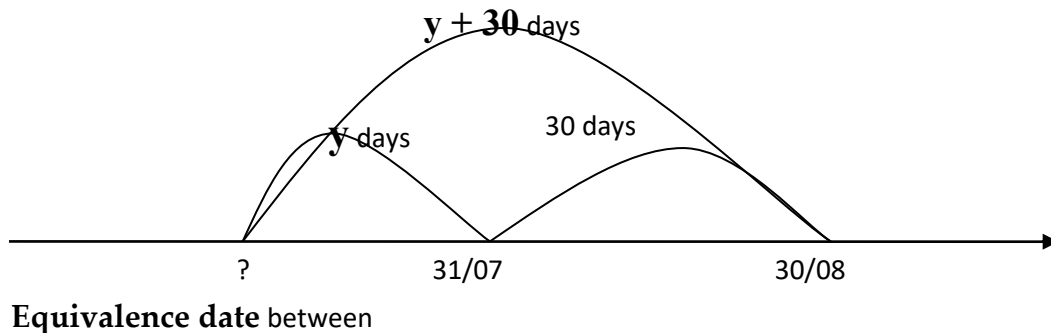
Maturity Date: 08/30

The discount rate is 4,5%.

Determine the equivalence date between the two instruments.

Notes:

1. If the equivalence date exists, it must be before the maturity date of the first paper.
2. The present values of the two instruments must be equal at the equivalence date



At the equivalence date

$$\frac{FV_1}{FV_2} = \frac{(D_{EV} - t_2)}{(D_{EV} - t_1)}$$

$$D_{EV} = \frac{36000}{4,5} = \frac{36000}{4,5} = 8000$$

$$\frac{1586,40}{1592,40} = \frac{(8000 - (y + 30))}{(8000 - y)}$$

$$1592,40 (7970 - y) = 1586,40 (8000 - y)$$

$$1592,40 \times 7970 - 1592,40 y = 1586,40 \times 8000 - 1586,40 y$$

$$228 = 6 y$$

$$y = 38 \text{ days}$$

The equivalence date is 06/23

Equivalence of a Single Commercial Instrument with Multiple Instruments

The equivalence of a single commercial instrument with several commercial instruments is based on the same principle used in the equivalence of two instruments, even when the number of instruments changes. According to this principle, the present value of the equivalent instrument is equal to the sum of the present values of the other instruments. From this relationship, the face value of the equivalent instrument (FV) can be determined as follows:

$$fV = \frac{fv_1(D_{EV} - t_1) + fv_2(D_{EV} - t_2) + fv_3(D_{EV} - t_3) + \dots + fv_n(D_{EV} - t_n)}{D_{EV} - t}$$

Where:

fV : Face value of the equivalent commercial instrument

$fv_1, fv_2, \dots, fv_n$: Face values of the individual commercial instruments

$t_1, t_2, \dots, t_n$: Times (in days) from the equivalence date to the maturity of each instrument

t : Time (in days) from the equivalence date to the maturity of the equivalent instrument

D_{EV} : *Devisor* which is $36000 / d$

Example

Three commercial instruments with the following maturity values and maturity dates are given:

1000 MU maturing on May 31.

4000 MU maturing on June 30.

3200 MU maturing on July 30.

The holder wishes to replace these three instruments with a single instrument maturing on June 20. The valuation is based on May 1 as the reference (equivalence) date.

The discount rate is **6% per annum**.

Required:

Determine the maturity value of the single equivalent instrument.

Solution:

$$D_{EV} = \frac{36000}{6} = 6000, \quad t = 50, \quad t_1 = 30, \quad t_2 = 60, \quad t_3 = 90$$

$$fV = \frac{fv_1(D_{EV} - t_1) + fv_2(D_{EV} - t_2) + fv_3(D_{EV} - t_3)}{D_{EV} - t}$$

$$fv = \frac{1000(6000 - 30) + 4000(6000 - 60) + 3200(6000 - 90)}{6000 - 50}$$

$$fv = \frac{5970000 + 23760000 + 189120000}{5950}$$

$$fv = 8175,12 \text{ MU}$$

Tutorial Sheet n° 04: Capital Equivalence under Simple Interest

Exercise 1

Determine the equivalence date for the following two commercial instruments at a discount rate of 9%:

- The first instrument has a maturity value of 7,800 MU due on May 31,
- The second instrument has a maturity value of 7,880 MU due on July 10.

Exercise 2

On August 22, a commercial instrument with a maturity value of 8,700 MU is negotiated, due on October 15. The discount terms are as follows: discount rate = 9.60%, duration-dependent commission rate = 0.60% , and duration-independent commission rate = 0.20%.

1. Calculate *the total banking fees excluding taxes* (agios excluding taxes) and the net amount from the negotiation (Proceeds or present value).
2. Calculate the effective discount rate **EDR**.
3. Calculate **the rate of return** in the discount operation.

Exercise 3

Complete the duplicate of the discount note that was improperly reproduced.
Discount note: delivered
on January 15

	cities	maturity Dates	Days	Discount	Collection commissions	
					Rate	Amount
7.260,00	Oran		12	21,78	free	Free
3.360,00	Tlemcen	31 January				6,72
22.920,00	Tiaret	16 February			1/2‰	
.....	Saida					
.....						20,66
329,82						
Net 38.170,18						

Discount at
 Commission - independent of duration 1‰
 Collection commissions 20.60

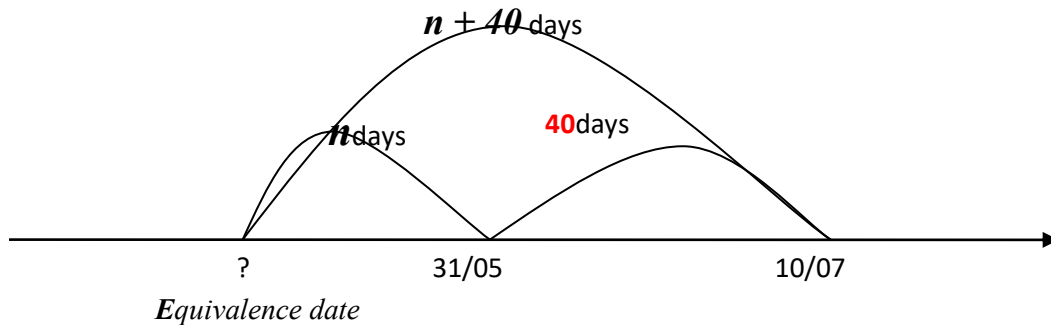
329.82

The collection commission is assumed to be independent of duration.

Solutions

Exercise 1

Let's define the sought equivalence date as being n days before May 31. Since the present values of the two instruments must be equal on the equivalence date, we can express this relationship as follows:



$$\begin{aligned}
 PV_1 &= PV_2 \\
 FV_1 - D_1 &= FV_2 - D_2 \\
 7800 - \frac{7800 \times 9 \times n}{36000} &= 7880 - \frac{7880 \times 9 \times (n + 40)}{36000}
 \end{aligned}$$

By solving the equation, we obtain the value of n .

$$80n = 4.800$$

$$n = 60 \text{ days}$$

The number of days between the two dates is 40 days. This gives:

$n = 60$ days before May 31, **which is April 1**

Exercise 2

1 – Calculation of the total **fees excluding taxes** and the net amount of the negotiation

- Total fees excluding taxes (Agiors hors taxes)

$$\begin{aligned}
 \text{Agios excluding taxes} &= \frac{FVd^t}{36000} + \frac{FVd^t}{36000} + \frac{FV_k}{100} \\
 \frac{8.700 \times 9,6 \times 54}{36.000} &+ \frac{8.700 \times 0,6 \times 54}{36.000} + (8.700 \times 0,2\%) \\
 &= 150,51 \text{ U/M}
 \end{aligned}$$

$$= 150.51 \text{ MU}$$

- Net amount of the negotiation: **Proceeds or present value**

$$P = FV - \text{Agios}$$

$$P = 8,700 - 150.51 = 8,549.79$$

$$= 8,549.79 \text{ MU}$$

2 - Calculation of the effective discount rate EDR

The effective discount rate **EDR** can be calculated in two ways:

First method:

$$EDR = \left[d + d' + \frac{360k}{t} \right]$$

$$9,6 \% + 0,6 \% + (360/54) \times 0,2\% = 11,53 \%$$

$$EDR = 11,53\%$$

Second method:

$$Agios = \frac{FV.t.(EDR)}{36000}$$

Therefore:

$$EDR = \frac{36000 \times agios}{fvxt}$$

$$EDR = \frac{36.000 \times 150.51}{8.700 \times 54}$$

$$EDR = 11.53\%$$

3 - Calculation of the cost rate

$$costrate = \frac{365 Agios}{PV \times t} = \frac{365 \times 150.51}{8.549,79 \times 54}$$

$$Cost rate = 11,8989 \text{ rounded to } 11,90\%$$

Exercise 3

principal	cities	Maturity date	days		Collection commission	
					Rate	Amount
7.260,00	Oran	January 27	12	21,78	free	Free
3.360,00	Tlemcen	January 31	16	13,44	2 ‰	6,72
22.920,00	Tiaret	february 16	32	183,36	1/2 ‰	11,46
4.960,00	Saida	february 26	42	52,08	1/2 ‰	2,48
38.500				270, 66		20,66
- 329,82						
=Net 38.170,18						

Clarification of the calculation process

Discount at 9%: 270.66

Commission independent of the duration: 1 ‰: 38.50

Collection commission: 20.66
 Total: 329.82

- We start by calculating the total of the discounts:

$38.170,18 + 329,82 = 38.500$. Then, we determine the value of the 4th note:

$38,500 - 7,260 - 3,360 - 22,920 = \mathbf{4,960}$.

- We calculate the commission independent of the duration:

$1\text{‰} = 38,500 \times 1\text{‰} = 38.50$.

- We calculate the total discount:

$329.82 - 20.66 - 38.50 = 270.66$.

From the first note, we determine the discount rate d :

$$d = \frac{36.000 \cdot D}{fv \cdot t} = \frac{36.000 \times 21,78}{7.260 \times 12} = \mathbf{9\%}$$

Date of submission: January 15. Therefore, the maturity date of the 1st note is January 27, which is $15 + 12$. From the submission date, it is also possible to determine t for the next two notes: 16 days for the 2nd and 32 days for the 3rd.

- The discount for the 4th note can be determined by subtraction:

$270.66 - 21.78 - 13.44 - 183.36 = 52.082$

From this, we can determine t :

$t = 36.000 \times 52,08 / 4.960 \times 9 = \mathbf{42 \text{ days}}$

The maturity date will be February 26.

Collection commissions: For the 2nd note, the rate is determined by dividing: $6.72 / 3,360 = 2\text{‰}$

For the 3rd note, we calculate the amount:

$1/2\text{‰} \times 22,920 = 11.46$.

For the 4th, we first calculate the amount: $20.66 - 6.72 - 11.46 = 2.48$.

Then, we calculate the rate: $2.48 / 4,960 = 1/2\text{‰}$.

Chapter 04

Compound interest

Concepts and Calculation of Compound Interest

Simple interest is typically calculated on short-term loans, generally for a period not exceeding one year, based on the principal amount for the entire duration. In contrast, compound interest for a given period is calculated based on the principal amount plus accumulated interest from previous periods. At the end of each period, the interest is added to the principal, forming a new basis for calculating interest in the following period—a process known as *capitalization*.

Compound interest is commonly used in cases where the principal is invested over multiple periods or several years, which is typical for medium- or long-term investments.

Key points

The interest may be compounded monthly, quarterly, semiannually, or annually.

Example:

Assume we deposited an amount of **2000 DZD** in a bank at an interest rate of **4%** for **4 years**. The table below shows the interest calculation based on both simple and compound interest to demonstrate the difference between the two methods.

Years	Simple interest		Compound interest		Future value
	Amount	Interest	Amount	Interest	
Year 1	2000	80	2000	80	$fv_1: 2000 + 80$
Year 2	2000	80	2080	83,2	$fv_2: 2080 + 83,2$
Year 3	2000	80	2163,2	86,53	$fv_3: 2163,2 + 86,53$
Year 4	2000	80	2249,73	89,99	$fv_4: 2249,73 + 89,99$

The accumulated value (principal plus interest) at the end of each period is capitalized and becomes the new principal for the next period. This process generates a sequence of future values, leading to the final future value at the end of the investment horizon.

1- Calculation of the Future Value or compound Amount (fv or A) : The future value refers to the principal amount invested plus the accumulated compound interest, and it is calculated using the following formula:

$$A = fv = p(1 + r)^n$$

Proof

Consider the following:

P: the principal amount

n = number of periods

r: the compound interest rate per period; and

FV: or A the future value or compound amount.

$$fv_1 = P + Pr = P \cdot (1+r)$$

$$fv_2 = fv_1 + fv_1 \cdot r = fv_1 \cdot (1+r) = P \cdot (1+r)(1+r) = P (1+r)^2$$

$$fv_3 = fv_2 + fv_2 \cdot r = fv_2 \cdot (1+r) = P (1+r)^2 \cdot (1+r) = P (1+r)^3$$

• •

• •

• •

$$fv_n = fv_{n-1} + fv_{n-1} \cdot r = fv_{n-1} (1+r) = P (1+r)^{n-1} \cdot (1+r) = P (1+r)^n$$

Note :

The future value (FV) is also referred to as the **accumulated value** in Anglo-Saxon financial terminology, and as the **acquired value (valeur acquise)** in French financial mathematics.

2- Calculation of Compound Interest Compound interest is calculated using the following formula:

$$I = P [(1+r)^n - 1]$$

Proof

We know that:

$$fv = I + P$$

$$I = fv - P \Rightarrow I = P (1+r)^n - P$$

$$\Rightarrow I = P [(1+r)^n - 1]$$

Example

A person deposited an amount of **8000 DZD** in the National Bank of Algeria at a compound interest rate of **5%** for **three years**.

- Calculate the **compound amount (future value)** obtained at the end of the deposit period.
- Calculate the total interest earned.

Solution

1. Calculation of the **compound amount** obtained at the end of the period:

$$fv = p (1 + r)^n$$

Thus:

$$fv_3 = 8000 \cdot (1 + 0.05)^3 = 9261 \text{ DZD}$$

2. Calculation of the interest:

$$I = P [(1+r)^n - 1]$$

$$I = 8000 [(1+0.05)^3 - 1] = 1261 \text{ DZD}$$

3- Calculation of the principal amount P (also known as the present value)

The present value is defined as the original amount, which can be determined from the *future value (compound amount)* at the end of its investment period. Thus, the present value is determined by subtracting the compound interest from the future value obtained at the end of the investment period. The principal is calculated using the formula:

$$P = fv (1 + r)^{-n}$$

Proof

If we have the following data:

- fv : the future value of a specific principal
- P : the principal amount
- n : number of periods
- r : interest compound rate

we know that:

$$fv = p (1 + r)^n$$

Therefore

$$P = \frac{fv}{(1 + r)^n} = fv(1 + r)^{-n}$$

$$P = fv(1 + r)^{-n}$$

Example

A person, X, deposits an amount in the BNA bank at an **8%** compound interest rate. After **5 years**, the account balance is **7540 DZD**.

Calculate the principal amount deposited in the bank

Solution

We have:

$$P = fv(1 + r)^{-n}$$

$$P = 7540 (1 + 0.08)^{-5} = 5131.60 \text{ DZD}$$

4- Calculation the Investment Period

This situation occurs when the accumulated amount from an invested sum is known, along with the compound interest rate, but the investment duration, t , is unknown.

Example

- what is the time period required for a principal of 40,000 DZD invested at a 5.5% interest rate to reach an amount of 44,000 DZD?
- How long will it take for the amount to double?

Solution

Given:

- Principal (P) = 40,000 DZD
- Final amount (A) = future value = 44,000 DZD
- Interest rate (r) = 5.5% = 0.055 (as a decimal)
- We need to calculate the time period (t) for the investment to grow to 44,000 DZD.

The formula for compound interest is:

$$fv = p(1 + r)^n$$

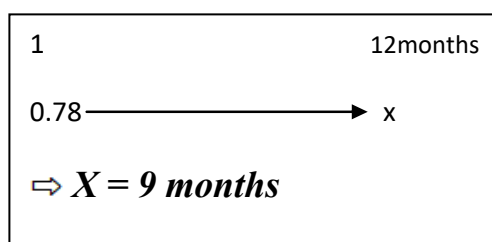
thus

$$44000 = 40000 (1 + 0.055)^n$$

$$\frac{44000}{40000} = 1.055^n \Rightarrow 1.1 = 1.055^n \Rightarrow \log(1.1) = \log(1.055^n)$$

$$\text{Therefore: } \log(1.1) = n \log(1.055) \Rightarrow n = \frac{\log(1.1)}{\log(1.055)} = 1.78$$

$$n = 1.78$$



That is, 1 year and 9 months.

Calculating the time needed for the principal to double

When the principal doubles at the end of the investment period, we have:

$$fv = 2P$$

On the other hand, we know:

$$fv = p(1 + r)^n$$

Thus:

$$2P = p(1 + r)^n$$

Therefore:

$$2 \times 40.000 = 40.0000(1 + 0.055)^n$$

$$80.000/40.000 = (1.055)^n$$

$$\text{Log } 2 = \log (1.055)^n$$

$$\text{Log } 2 = n \cdot \log (1.055)$$

Therefore

$$= n \frac{\log 2}{\log 1.055}$$

$$n = 12,95$$

1 year...12 months

0.95.....11 months

Since 1 year = 12 months:

0.95 years $\approx$ 11 months.

Thus, the time t required for the principal to double is approximately **12 years and 11 months.**

Equivalent Rates in Compound Interest

In the context of compound interest, **equivalent rates** refer to interest rates that, when applied over different compounding periods, yield the same future value (accumulated value) for an investment at the end of the investment period.

When an interest rate is compounded more frequently than annually (for example, quarterly, semi-annually, or monthly), the interest for each period is calculated on the new balance (including the accumulated interest). However, we might want to compare these rates to an **annual nominal rate** (an annual rate applied over a year) to maintain consistency in comparisons. The **equivalent rate** is a rate that, if compounded in a different period (e.g., monthly or quarterly), would result in the same future value as an annual rate compounded once at the end of the year.

Key points

An equivalent interest rate is an interest rate that produces the same future value as another interest rate, even when the compounding frequency differs, provided that the principal and the investment period remain unchanged.

Formula for Equivalent Rates

For an annual nominal interest rate compounded annually, the equivalent interest rate for a different compounding period (such as monthly or quarterly) is determined by ensuring the same future value for the same principal and the same investment horizon, despite differences in compounding frequency. This approach allows for the comparison of interest rates expressed over different compounding periods on a basis of financial equivalence.

A distinction can be made between two cases depending on the direction of conversion: from an annual rate to a periodic rate, or from a periodic rate to an annual rate.

1. **From annual to periodic rate** (e.g., quarterly, semi-annual, or monthly):

If r is the annual nominal rate, and k is the number of compounding periods per year, the **equivalent rate for each period (r_k)** is given by:

$$= (1 + r)^{1/k} - 1r_k$$

Where:

- r_k : Is the rate for each period (quarterly, semi-annually, etc.),
- r : is the annual nominal rate (expressed as a decimal),
- k : is the number of compounding periods per year (e.g., for monthly compounding, $k = 12$; for quarterly, $k = 4$).

2. From periodic to annual rate:

If the interest is compounded periodically (e.g., quarterly), and you want to find the equivalent annual rate r (annual nominal rate), you can use:

$$r = (1 + r_k)^k - 1$$

Where:

- r_k : is the rate per period (monthly, quarterly, etc.),
- r : is the equivalent annual rate.

Proof of the Equivalent Rate formula (periodic rate)

We start from the equation for compound interest:

$$fv = p(1 + r)^n = p(1 + r_k)^{nk}$$

Taking both sides of the equation to the power of $1/n$ to eliminate n , we get:

$$[P(1 + r)^n]^{\frac{1}{n}} = [P(1 + r_k)^{nk}]^{\frac{1}{n}}$$

$$(1 + r) = (1 + r_k)^k$$

Taking both sides to the power of $1/k$ to eliminate k on one side. We then obtain:

$$[1 + r]^{\frac{1}{k}} = [(1 + r_k)^k]^{\frac{1}{k}}$$

$$(1 + r)^{\frac{1}{k}} = (1 + r_k)$$

Next, we move 1 to the other side, we get:

$$\rightarrow r_k = (1 + r)^{\frac{1}{k}} - 1$$

Example

If the annual interest rate is $r=8\%$ (0.08), the equivalent monthly interest rate can be determined by assuming monthly compounding.

$$\begin{aligned}
 &= (1 + r)^{1/k} - 1r_k \\
 &= (1 + 0.08)^{1/12} - 1r_k \\
 &= 0.006434 \text{ or } 0.6434\%
 \end{aligned}$$

This means the monthly equivalent rate for an 8% annual nominal rate is approximately 0.6434% per month.

Example

Calculate the following equivalent rates: the semi-annual, quarterly, and monthly equivalent rates for an *annual nominal rate* of 9%.

Solution

$$= (1 + r)^{1/2} - 1r_k$$

Semi-annual equivalent rate r_2 :

$$(1 + 0.09)^{1/2} - 1 \text{ so } r_2 = 0.044 * 100$$

$$\text{Therefore: } r_2 = 4.4\%$$

Quarterly equivalent rate r_4 :

$$r_4 = (1 + r)^{1/4} - 1 \text{ therefore: } r_4 = (1 + 0.09)^{1/4} - 1 = 0.022$$

The quarterly rate r_4 is:

$$r_4 = 2.2 \%$$

Monthly equivalent rate r_{12} :

$$r_{12} = (1 + r)^{1/12} - 1 \text{ so } r_{12} = (1 + 0.09)^{1/12} - 1 = 0.0072$$

$$r_{12} = 0.72 \%$$

Discounting at Compound Interest

Discounting at compound interest is the reverse process of capitalization. Capitalization determines the future value of a present sum, while discounting determines the present value of a future sum by finding the amount that would grow to it under compound interest.

It is used to evaluate the present value of future cash flows such as investments, bonds, and loans.

Under compound interest, the discount is the difference between the nominal value and the present value, which is obtained by discounting future cash flows at a compound interest rate over the remaining time to maturity

Key Points

Compound interest discounting is the reverse process of capitalization, used to determine the present value of future amounts. It is mainly applied to the valuation of future cash flows such as investments, loans, and bonds.

Present Value Formula

At compound interest, the relationship between the **present value (P)**, the **future value (fv or A)**, the **interest rate (r)**, and the **number of periods (n)** is given by:

$$A = fv = p(1 + r)^n$$

When the present value is required, this relationship is rearranged as:

$$P = fv(1 + r)^{-n}$$

Interpretation

The term: $v^n = \frac{1}{(1+r)^n}$ is called the **compound discount factor** (or **present value factor**). It represents the present value of one monetary unit payable after n periods at an effective interest rate.

In other words, it is the amount that must be invested today at a compound interest rate r for n periods to accumulate to one monetary unit in the future.

Example

Find the present value of DZD 10,000 due in 3 years if the annual compound interest rate is 8%.

$$P = 10,000(1 + 0.08)^{-3} = 7,943.28 \text{ DZD}$$

Interpretation:

An amount of DZD 7,943.28 invested today at 8% compound interest will become DZD 10,000 after 3 years.

Discount formula

$$D = fv - p$$

$$D = fv - fv(1 + r)^{-n}$$

$$D = fv(1 - (1 + r)^{-n})$$

Example

A financial instrument with a nominal value of **750,000 DZD** is due in **5 years**. Determine the **present value** and the **compound discount** at an annual interest rate of **5%**.

Solution

Present value:

$$P = fv(1 + r)^{-n}$$

$$P = 750000(1 + 0.05)^{-5}$$

$$P = 588,020.46 \text{ DZD}$$

Discount:

$$D = fv - pv$$

$$D = 750\,000,00 - 588\,020,46 \approx 161,979.54 \text{ DZD}$$

Proportional Rates VS Equivalent Rates

In financial mathematics, interest rates play a critical role in determining the value of money over time. One type of interest rate used in certain financial transactions is the **proportional rate**.

What are Proportional Rates?

Proportional rates, also known as **simple interest rates**, are used to calculate interest over a period shorter than a year. These rates are proportionally divided according to the time period used for interest calculation, such as semi-annual, quarterly, or monthly.

Definition: Two rates are said to be **proportional** if their ratio is equal to the ratio of their corresponding time periods. Both rates will result in the same accumulated value after the same total time, but they are based on different compounding periods.

Proportional Rate formula

If the year is divided into **k** periods (e.g., 2 for semi-annual, 4 for quarterly), the **proportional rate** is calculated by dividing the **annual rate** by the number of periods in a year:

$$\text{Proportional Rate} = \text{annual rate} / k$$

Key Features of Proportional Rates

- **Simple Interest:** Proportional rates are typically used for **simple interest** calculations, where interest is calculated only on the initial principal amount.
- **Short-Term Financial Transactions:** Proportional rates are most often used in **short-term investments** or loans, where the interest period is shorter than one year.

Example 1

Proportional Rate for Semi-Annual Compounding

If the annual interest rate is **10%** and we need to calculate the **semi-annual** proportional rate, we simply divide the annual rate by 2 (because there are 2 semi-annual periods in one year):

$$\text{Semi-annual Proportional Rate} = 10\% / 2 = 5\%$$

Example 2**Proportional Rate for Quarterly Compounding**

For **quarterly** compounding, where the year is divided into 4 periods:

Quarterly Proportional Rate = $10\%/4 = 2.5\%$

Quarterly Proportional Rate = $4 \times 10\% = 2.5\%$

When to Use Proportional Rates ?

Proportional rates are primarily used when calculating **simple interest** for short-term financial products. They are ideal when the interest calculation does not need to account for the compounding effect and is based on a fixed rate over shorter periods.

Common Use Cases:

1. **Short-term loans:** Personal loans or small business loans where interest is calculated using a fixed annual rate but applied to shorter periods (e.g., monthly or quarterly).
2. **Investments with simple interest:** Investments where interest is applied based on the initial investment without compounding.

*Comparison of Proportional and Equivalent Interest Rates***Example 3**

Prepare a table showing the **proportional rates** (semi-annual, quarterly, and monthly) for an **8%** annual rate, in both **simple interest** and **compound interest** systems, and the accumulated values for each case with an initial investment of **1000 DZD** for period of one year.

Solution

<i>period</i>	<i>Proportional Rate Simple Interest</i>	<i>Equivalent Rates Compound Interest</i>	<i>Accumulated Value (Simple Interest)</i>	<i>Accumulated Value (Compound Interest)</i>
<i>Annual rate</i>	8%	8%	$1000 + 1000 \times 0.08 \times 1 =$ 1080DA I=80DA	$1000(1.08) =$ 1080DA I=80DA
<i>Semi-annual rate</i>	$8\% / 2$ 4%	$= (1 + r)^{1/k} - r_k$ 1 $- 1(1 + 0.08)^{1/2}$ $= 0.0392$ 3.92%	$1000 + 1000 \times 0.04 \times 2 =$ 1080DA I=80DA	$1000(1.0392)^2 =$ 1080DA I=80 DA
<i>Quarterly Rate</i>	$8\% / 4$ 2%	%1.94	$1000 + 1000 \times 0.02 \times 4 =$ 1080DA I=80DA	$1000(1.0194)^4 =$ 1080DA I=80 DA
<i>Monthly Rate</i>	$8\%/12$ 0.67%	%0.64	$1000 + 1000 \times 0.0066 \times 12 =$ 1080DA I=80DA	$1000(1.0064)^{12} =$ 1080 DA I=80 DA

This demonstrates that while the rates differ slightly between the simple and compound systems, the accumulated values for one year remain consistent in this scenario

Differences between Proportional Rates and Equivalent Rates

Criterion	<i>Proportional Rates</i>	<i>Equivalent Rates</i>
Definition	Rates obtained by dividing the annual rate by the number of compounding periods.	Rates that produce the same accumulated value over a given period.
Interest System	Primarily associated with simple interest or nominal rate conversions.	Associated with compound interest.
Calculation Method	$\text{Proportional Rate} = \text{annual rate} / k$	$r_k = (1 + r)^{1/k} - 1$
Interest Accumulation	Does not consider interest on previously earned interest.	Considers interest earned on previously accumulated interest.
Purpose	To express an annual rate over shorter periods (monthly, quarterly, etc.).	To determine rates that are financially equivalent despite different compounding frequencies.
Typical Use	Short-term financial calculations and nominal rate expressions.	Long-term financial operations involving compound interest.

Equivalence of capitals under compound interest

Equivalence does not mean that the two capitals are equal in nominal value, but that they have the **same financial value at a given date** when the time value of money is taken into account.

Equivalence of capitals under compound interest differs from simple interest in that it remains valid regardless of the date at which it is established. In fact, if two or more present values are equal at a given moment, this equality will also hold at any other date, even though the corresponding values will change over time.

1. Equivalence of Two Capitals

Under compound interest, the equivalence of two capitals is determined by comparing their present values at the same reference date, using a single discount rate. Two capitals are said to be equivalent if their present values, calculated with the same discount rate i , are equal at the same point in time, regardless of differences in their maturity date

Let us consider the following data for two commercial instruments:

1) First Commercial Instrument

- a. Maturity Value (face value): FV_1
- b. Present Value (Proceeds): PV_1

2) Second Commercial Instrument

- a. Maturity Value (face value): FV_2
- b. Present Value (Proceeds): PV_2

n_1 and n_2 : the number of periods after which the two capitals are due

When equivalence is established at the initial time (time zero), the condition of equivalence is expressed by the equality of present values at that date. Consequently, the following formula holds:

$$Pv_1 = Pv_2$$

$$Fv_1(1 + r)^{-n_1} = Fv_2(1 + r)^{-n_2}$$

2. Equivalence of a Single Capital and a Set of Capitals

A single capital is said to be equivalent to a set of several capitals if, at any chosen date, its present value is equal to the sum of the present values of all the capitals in the set.

Let:

- **Fv** : the face value of the single capital;
- **n** : the number of periods until the single capital is payable;
- **Fv_k** : the face value of the capital in the set, $k=1, \dots, k=s$;
- **n_k** : the number of periods until the k capital is payable;
- **r** : the valuation rate (discount rate).

The equivalence at time zero between the single capital and the set of capitals is expressed as:

$$fv(1+r)^{-n} = \sum_{k=1}^{k=s} Fv_k(1+r)^{-n_k}$$

Example:

At an annual interest rate of **9%**, a company wants to replace several outstanding debts with a **single equivalent payment** due at the end of year 8.

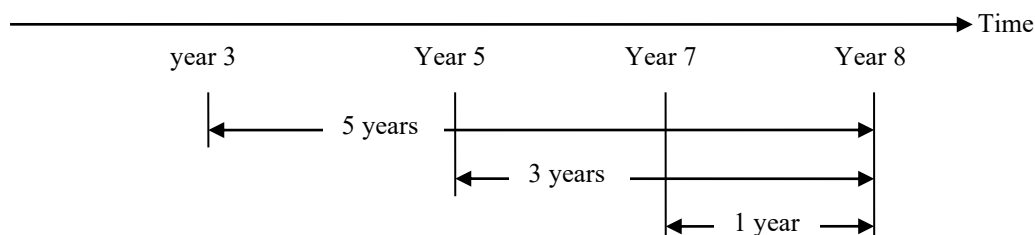
The existing obligations are:

- 1,500 DZD due at the end of year 3
- 2,800 DZD due at the end of year 5
- 3,500 DZD due at the end of year 7

Required:

Determine the **nominal value of the single payment due at the end of year 8** that is financially equivalent to these debts.

solution



$$V(8)=1500(1.09)^5+2800(1.09)^3+3500(1.09)^1= \mathbf{9753.95 \text{ DZD}}$$

The three original debts are replaced by **one single payment of approximately 9,753.95 DZD at the end of year 8.**

Tutorial Sheet n° 05: Compound Interest and Compound Discounting**Exercise 1**

Complete the table below:

Duration	Rate	principal (DA)	Future Value (DA)
.....	7% per year	22.500	50.674,31
20 years	3% per semestre	6.000	
.....	7,5% per year	17.000	20.004,05
5 years 9months		20.000	29.807,23

Exercise 2

A principal of 15,000 DA is placed for 4 years at an annual rate of 9%. Calculate its future value (acquired value) if the capitalization period is:

- Semi-annual;
- Quarterly;
 1. Using proportional rates;
 2. Using equivalent rates.

Exercise 3

Calculate the present values of the following investments:

- 22,000 DA payable in 5 years and 5 months, rate 11% per year;
- 15,000 DA payable in 6 years, rate 9.25% per year;
- 18,000 DA payable in 13 years, 2 quarters, and 2 months at 9% per year.

Exercise 4

How long will it take for an amount placed at compound interest, at an annual rate of 7.5%, to double?

Exercise 5

At what annual compound interest rate should a principal be invested to triple its value in 9 years?

Exercise 6

A person can invest an amount of money using two (02) investment options over (n) years:

- Option A: 7.5% per year with compound interest;
 - Option B: 9.3% per year with simple interest.
1. What should they choose if $n=6$?
 2. What should they choose if $n=7$?
 3. What is approximately the value of n for which both options are equivalent?

Solutions

Exercise 1

	Duration	Rate	Capital (DA)	Valeur acquise (DA)
1	12 years	7% per year	22.500	50.674,31
2	20 years	3 per semester	6.000	19.572,23
3	9 trimestres	7,5% per year	17.000	20.004,05
4	5 years and 9 months	1,75% per trimester	20.000	29.807,23

1) Rate 7% per year

$$\text{Duration} = ? \rightarrow fv_{12} = 22.500(1,07)^n = 50.674,31$$

$$p = 22.500 \text{ DA}$$

$$fv = 50.674,31$$

$$n = \frac{\log(50.674,31/22.500)}{\log(1,07)} = \mathbf{12}$$

2) rate = 3% per semestre

$$\text{Duration} = 20 \text{ years} = 20 \times 2 = 40 \text{ semestres}$$

$$p = 6.000 \text{ DA}$$

$$fv_{40} = ? \text{ therefore } fv_{40} = 6.000(1,03)^{40}$$

$$\mathbf{fv_{40} = 19.572,23 \text{ DA}}$$

3) rate = 7,5% per year

$$\text{Duration } n \text{ per years}$$

$$p = 17.000 \text{ DA}$$

$$fv = 20.004,05 \text{ DA}$$

$$fv = 17.000 (1,075)^n = 20.004,05 \text{ so } (1,075)^n = 1,1767088$$

$$n = \frac{\log(1,1767088)}{\log(1,075)} = \mathbf{2,249999 \approx 2.25}$$

$$0.25 \text{-----} x$$

$$1 \text{ --- } 12 \text{ months therefore } x = 3 \text{ months } (0.25 * 12 = 3 \text{ months})$$

The period is therefore 2.25 years, or **9 quarters**.

4) Duration = 5 years and 9 months, or **23 quarters**.

$$p = 20.000 \text{ DA}$$

$$fv_{23} = 29.807,23 \text{ DA}$$

Rate = ?

Calculation of quarterly rate: r_k

$$20.000 (1 + r_k)^{23} = 29.807,23$$

$$(1 + r_k) = 1,4903615^{\frac{1}{23}}$$

$$r_k = 1,4903615^{1/23} - 1 \quad \text{therefore } r_k = 0,0175$$

So the quarterly rate is 1.75%.

Exercise 2

→ With proportional rates, a 9% annual rate corresponds to: : $9/2 =$
4,5% Semi-annual rate;

→ $r_2 = 4.5\%$, $k = 2$, $n = 4$ years

The future values are :

→ during $4 \times 2 =$ **8 semesters**

$$fv = p(1 + r_k)^{n \times k} \rightarrow$$

→

$$fv_8 = 15.000 (1,045)^8 = \mathbf{21.331,50 DA}$$

→ $9/4 =$ **2,25% Quarterly rate;**

→ During $4 \times 4 =$ **16 trimesters**

→ $r_2 = 2.25\%$, $k = 4$, $n = 4$ years

$$fv = p(1 + r_k)^{n \times k} \rightarrow$$

$$\rightarrow fv_{16} = 15.000 (1,0225)^{16} = \mathbf{21.414,32 DA}$$

1) With equivalent rates at 9%

$$r_k = (1 + r)^{1/k} - 1$$

→ $r_s = (1,09)^{\frac{1}{2}} - 1$ soit 4,40% per semester

→ $r_t = (1,09)^{1/4} - 1$ soit 2,17% per trimester

The two future values are :

→ During $4 \times 2 = 8$ semesters

$$fv_8 = 15.000 (1,044)^8 = \mathbf{21.168,75 DA}$$

→ During $4 \times 4 = 16$ trimesters

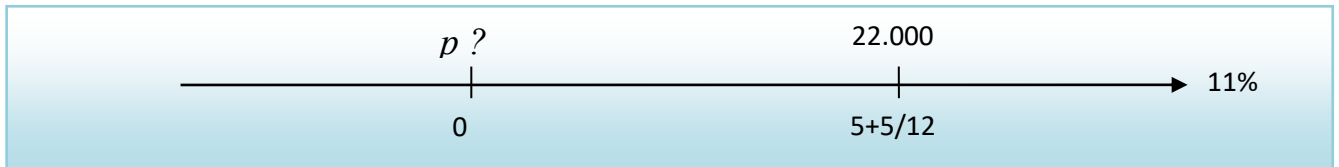
$$fv_{16} = 15.000 (1,0217)^{16} = \mathbf{21.147,81 DA}$$

Exercise 3

There is no indication regarding the rates.

Therefore, it is an annual rate.

The present value



$$P = fv(1 + r)^{-n}$$

$$p = 22.000 (1,11)^{-(5+\frac{5}{12})} \quad \text{so } p = \mathbf{12.500,38 \text{ DA}}$$



$$P = 15.000 (1,0925)^{-6} \quad \text{therefore } P = \mathbf{8.821,91 \text{ DA}}$$

$$p = 18.000 (1,09)^{-(13+\frac{8}{12})} \quad \text{therefore } p = \mathbf{5.543,41 \text{ DA}}$$

Exercise 4

If P is the amount invested,

n is the number of investment years.

$$\mathbf{P}(1,075)^n = 2 \quad \text{so } (1,075)^n = 2$$

$$n = \frac{\log 2}{\log(1,075)} = \frac{0,3010299}{0,0314084} = \mathbf{9,5843755 \text{ years}}$$

That is 9 years and 7 months.

Exercise 5

If r is the annual rate and P is the invested principal, then:

$$p(1 + r)^9 = 3p \quad \text{Which simplifies to:}$$

$$(1 + r)^9 = 3 \quad \text{thus } r = 3^{1/9} - 1$$

$$r = 0,129830$$

This corresponds to an approximate annual rate of 12.98%.

Exercise 6

Let P be the invested amount .If $n=6$ years:

→ Option A = $p(1,075)^6 = 1,54333$ pfv using compound interest

→ Option B = $p + \frac{px 9,3 \times 6}{100} = 1,5580p$ future value using compound interest

Choose Option B.

1) If $n=7$ years:

→ Option A = $p(1,075)^7 = 1,6590p$

→ Option B = $p + \frac{C_0 \times 9,3 \times 7}{100} = 1,651p$

Choose Option A

We know the equivalence occurs between 6 and 7 years. To find the point of equivalence, the following equality must be satisfied:

$$p(1,075)^n = p + [px 9,3 \times n]/100$$

$$p(1,075)^n = p[1 + 0,093 n]$$

$$(1,075)^n = 1 + 0,093n$$

This simplifies to:

$$(1,075)^n - 0,093n - 1 = 0$$

We know : $6 < n < 7$

Using linear interpolation:

6	→	-0,0146986
n	→	0
7	→	0,0080491

$$\frac{n - 6}{7 - 6} = \frac{0 + 0,0146986}{0,0080491 - 0,0146986} = 0,646156$$

So : $n = 6,46156$ years

This corresponds to 6 years, 7 months, and 23 days.

Chapter 05

Annuities

Annuities

Annuity Terminology and types

An annuity is a financial product that provides a series of regular payments made over a specified period of time. These payments can be made annually, semi-annually, quarterly, or monthly, depending on the terms of the agreement. Annuities are commonly used in both personal finance and business, often as a way to provide a steady stream of income, manage long-term investments, or fund retirement.

Annuities types

There are several types of annuities, each with its unique characteristics and applications:

When the annuity payments are equal, they are referred to as **level annuities** (or **fixed annuities**). When the payment amounts vary from one period to another, they are called **variable annuities**.

Annuities may be **due** (paid or received at the **beginning of each period**) or **ordinary** (paid or received at the **end of each period**).

Key Points :

An annuity is a series of regular payments made over a specified period of time. It may be fixed or variable, and payments may be made either at the beginning or at the end of each period.

Annuities can be classified as follows:

- **Certain annuities**, when the number of payments is known in advance;
- **Contingent annuities** (such as **life annuities**), when the number of payments is uncertain and depends on an event like the lifespan of an individual;
- **Perpetuities**, when the payments continue indefinitely.

Valuation of Annuities: Present and Future Value

- **The future value of an annuity** quantifies how much your periodic payments will be worth in the future. To achieve the overarching goal of having enough money to live comfortably in retirement, you want

the future value of your annuity to be worth more than its present value.

- **The present value** of an annuity is the total value today of a series of future periodic payments, discounted at a specific **discount rate**. In other words, it tells you how much you would need to invest right now to receive a set amount of money at regular intervals in the future. Common examples of present value annuities are loans and mortgages (bonds).

Present versus future value of an annuity

The future value of an annuity is worth more than its present value because, through additional contributions and interest earned, your annuity grows on a compounding basis.

Think of present value the opposite way: It determines how much money you must contribute today to realize a desired outcome tomorrow, factoring in a fixed (or anticipated, if a variable annuity) rate of return over the life of your annuity.

That's why the future value should always be worth more than the present value. So, if you want to have 6,500 DZD in 10 years (future value), you would need to deposit 5,000 DZD today (present value) and achieve an annual average rate of return of 5.5% to get there.

Note:

Throughout the study of annuities and loans, the interest rate may be denoted by either i or r , and the regular periodic payment may be denoted by either R or PMT , depending on the notation adopted.

- I- Ordinary Annuities (end of each period):** As sequence of equal payments made at equal periods of time is called an **annuity**. If the payments are made at the end of the time period, and if the frequency of payments is the same as the frequency of compounding, the annuity is called an **ordinary annuity**. The time between payments is the **payment period**, and the time from the

beginning of the first payment period to the end of the last period is called the **term** of the annuity.

Key Points :

Ordinary annuity is series of payments which possess the following characteristics:

- 1) Amount of payment should be same
- 2) Interval time between the payment should be same
- 3) Payment should be made at the end of each period

I.1. The future value of ordinary annuity:

Future value of ordinary annuity – refers to the value that is compounding till the end of its term/duration or it is also defined as the sum of future value of all the periodic payments at the end of annuity.

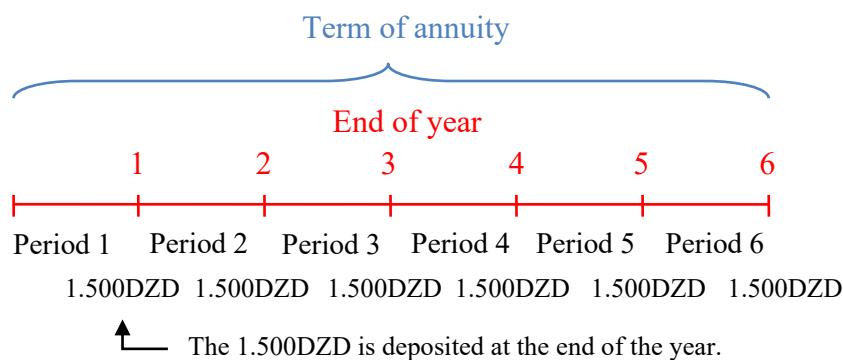
Example

Suppose 1500 DZD is deposited at the end of each year for the next 6 years in an account paying 8% per year compounded annually. How much will be in the account at the end of this period?

Solution

The following Figure shows this annuity. To find the future value of the annuity, look separately at each of the 1500 DZD payments. The first of these payments will produce a compound amount of

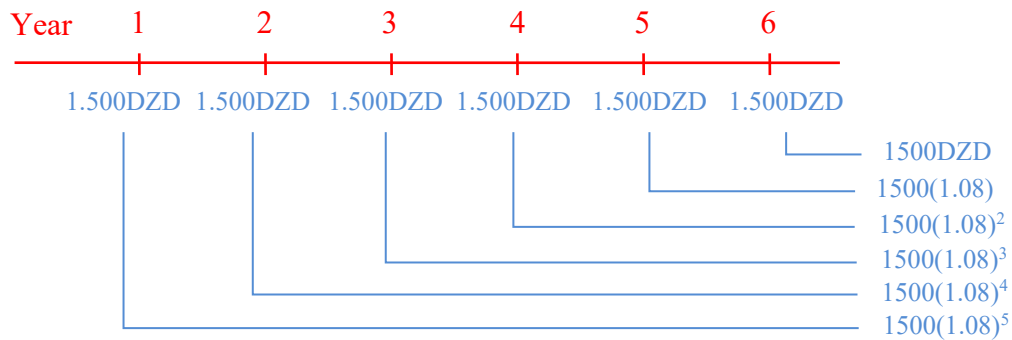
$$1500(1 + 0.08)^5 = 1500(1.08)^5$$



Use 5 as the exponent instead of 6, since the money is deposited at the *end* of the first year and earns interest for only 5 years. The second payment of 1500DZD will produce a compound amount of $1500(1.08)^4$. As shown in following Figure, the future value of the annuity is

$$1500(1.08)^5 + 1500(1.08)^4 + 1500(1.08)^3 + 1500(1.08)^2 + 1500(1.08)^1 + 1500$$

(The last payment earns no interest at all)



Reading this sum in reverse order, we see that it is the sum of the first six terms of ageometric sequence, with $a = 1500$, $r = 1.08$, and $n = 6$. Thus, the sum equals

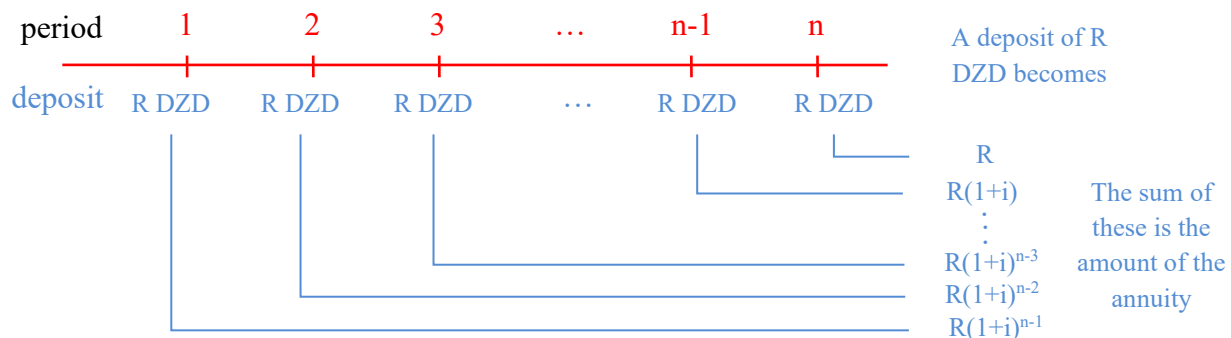
$$\frac{a(r^n - 1)}{r - 1} = \frac{1.500[(1.08)^6 - 1]}{1.08 - 1} \approx 11.003,89$$

To generalize this result, suppose that payments of R (*PMT in some references*) dinars each are deposited into an account at the end of each period for n periods, at a rate of interest i (*or r in some references*) per period. The first payment of R dinars will produce a compound amount of $R(1 + i)^{n-1}$ dinars, the second payment will produce $R(1 + i)^{n-2}$ dinars, and so on; the final payment earns no interest and contributes just R dinars to the total. If S represents the future value (or sum) of the annuity, then (as shown in Figure below),

$$FV = R(1 + i)^{n-1} + R(1 + i)^{n-2} + R(1 + i)^{n-3} + \dots + R(1 + i) + R$$

Or, written in reverse order,

$$FV = R + R(1 + i)^1 + R(1 + i)^2 + \dots + R(1 + i)^{n-1}$$



This result is the sum of the first n terms of the geometric sequence having first term R and common ratio $1 + i$. Using the formula for the sum of the first n terms of a geometric sequence,

$$FV = \frac{R[(1+i)^n - 1]}{(1+i) - 1} = \frac{R[(1+i)^n - 1]}{i} = R \left[\frac{(1+i)^n - 1}{i} \right]$$

A formula for the future value of an annuity S of n payments of R dinars each at the end of each consecutive interest period, with interest compounded at a rate i per period, follows. Recall that this type of annuity, with payments at the *end* of each time period, is called an ordinary annuity.

Formula for Future Value of an Ordinary Annuity

$$FV = R \left[\frac{(1+i)^n - 1}{i} \right]$$

Where:

- FV: is the future value;
- **R : Regular payment** is the periodic payment at the end of period (PMT);
- i : is the interest rate per period;
- n : is the number of periods.

I.2. The present value of ordinary annuity

The present value (PV) of an ordinary annuity is the current worth of a series of future, equal payments made at the end of each period, the formula for the Present value of an ordinary annuity is :

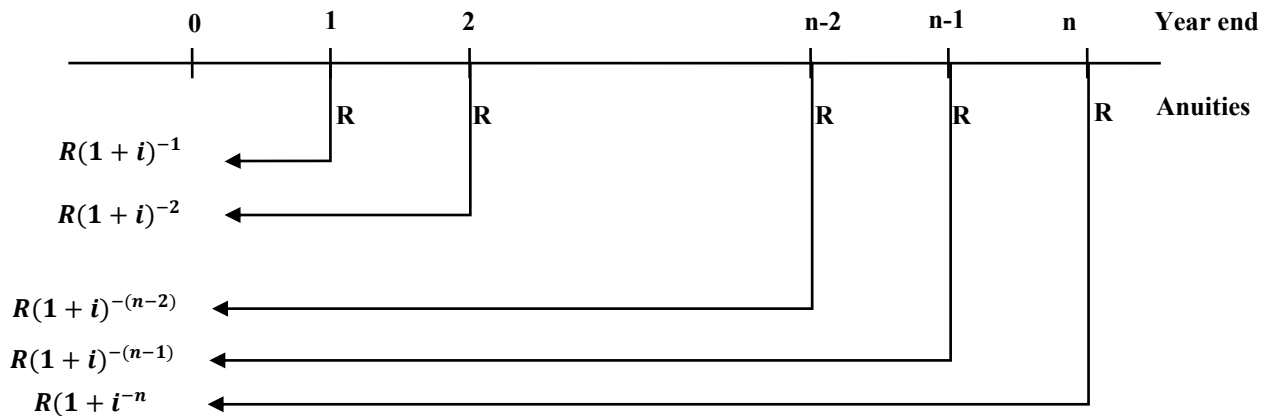
$$Pv = R \left[\frac{1 - (1+i)^{-n}}{i} \right]$$

Where :

- PV : Present Value of annuity stream
- R or PMT in some references Regular payment at the end of each payment interval
- i = interest rate per period
- n = number of period for which annuity will lasts

Derivation of present value annuity formula

Let's explore how the **present value annuity formula** is derived. Although this derivation is **not directly examinable**, it is important for students to **understand the reasoning and principles** behind the formula's development.



The 1st payment made at the end of 1st year discounts to $R(1+i)^{-1}$; the 2nd payment made at the end of 2nd year discounts to $R(1+i)^{-2}$; the (n-2)th payment made at the end of (n-2)th year discounts to $R(1+i)^{-(n-2)}$; (n-1)th payment made at the end of (n-1)th year discounts to $R(1+i)^{-(n-1)}$ and the last payment made at the end of the nth year is discounted by $R(1+i)^{-n}$ when brought to the beginning of the period.. The sum of all the discounted payments (denoted by P) is expressed as below:

$$P = R(1+i)^{-1} + R(1+i)^{-2} + \dots + R(1+i)^{-(n-2)} + R(1+i)^{-(n-1)} + R(1+i)^{-n}$$

This is a geometric series, which results in the following summation:

$$PV_n = R \left[\frac{1 - (1+i)^{-n}}{i} \right]$$

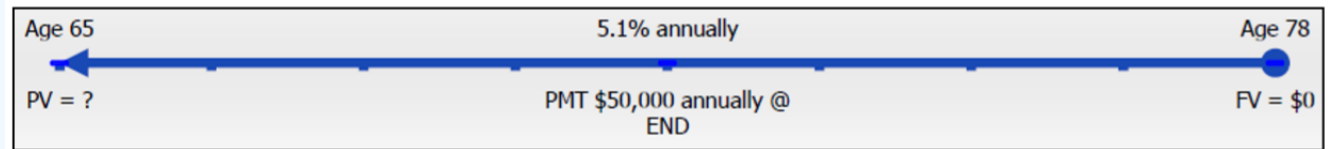
Example:

Mohamed is planning on having an annual gross income of 50 000 DZD at the end of every year when he retires at age 65. He is planning for the account to be emptied by age 78. If the account earns 5.1% compounded annually, what amount of funds needs to be in the account when he retires?

Solution

Step 1:

The payments occur at the end of each payment interval, and the compounding period is identical to the payment interval. Therefore, this is an **ordinary annuity**. The objective is to determine its value at the beginning of the term, which corresponds to its present value. The timeline of the client's account is shown below.

**Step 2:**

$R(\text{PMT}) = 50,000 \text{ DZD}$, $i = 5.1\%$, $n = 78 - 65 = 13$.

$$PV_n = R \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

$$PV_n = 50\,000 \left[\frac{1 - (1 + 0.051)^{-13}}{0.051} \right]$$

$$PV_n \approx 460\,685 \text{ DZD}$$

II- Annuity due (immediate annuity)

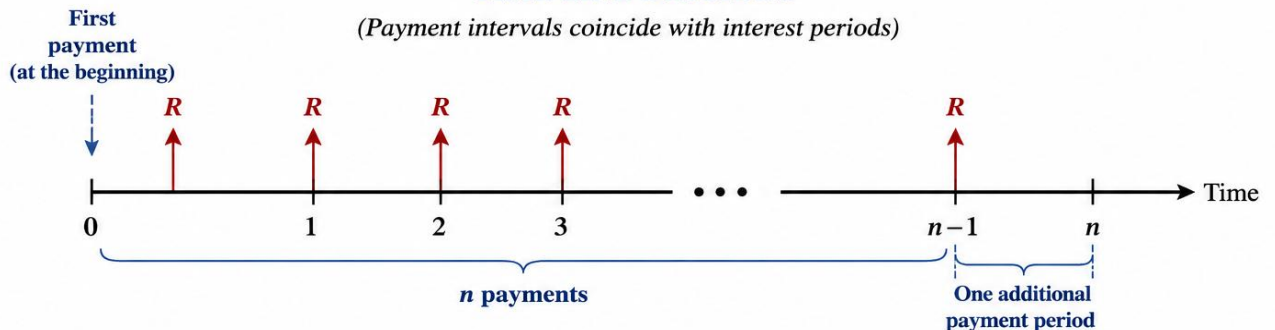
An annuity due is an annuity in which periodic payments are made at the beginning of each payment period. The term of an annuity due begins at the time of the first payment and ends one payment period after the last payment. The diagram below illustrates the simple case of an annuity due with n payments, assuming that the payment intervals coincide with the interest conversion periods. The annuity due can be represented as follows:

ANNUITY DUE (n PAYMENTS)

An annuity due is an annuity in which payments are made **at the beginning** of each payment period.

TIME LINE DIAGRAM

(Payment intervals coincide with interest periods)



The term of an annuity due begins at the time of the first payment (time 0) and ends **one payment period after the last payment** (time n).

KEY POINTS

- Payments are made at the **beginning** of each period.
- There are n payments in total.
- The last payment is made at time $n-1$.
- The term ends at time n .

PRESENT VALUE AT TIME 0

The present value of an annuity due is:

$$PV_{\text{due}} = R \left[\frac{1 - (1+i)^{-n}}{i} \right] (1+i)$$

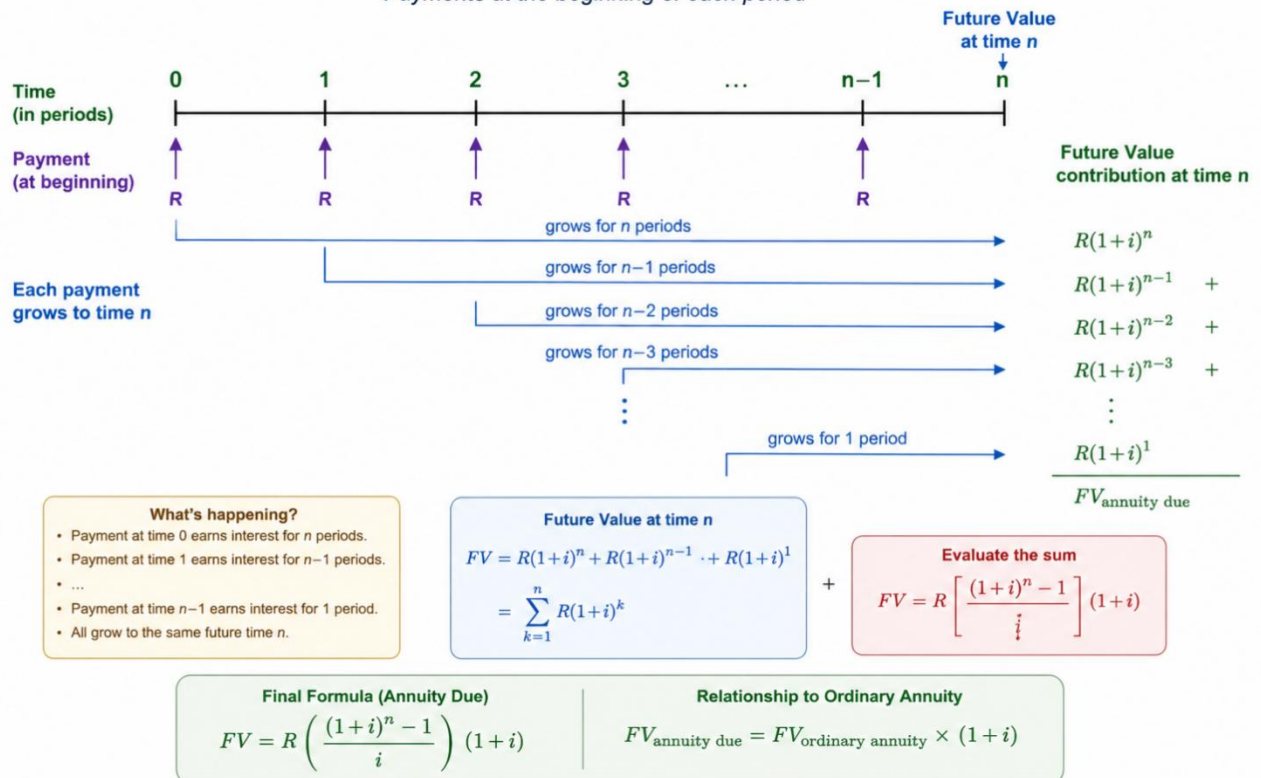
where R = payment per period, i = interest rate per period,
 n = number of payments.

II.1. Future Value of an Annuity Due

The future value of an annuity due exceeds that of an ordinary annuity since each payment is made at the beginning of the period and therefore earns interest for an additional compounding period. The diagram below presents the timing of payments R and demonstrates the step-by-step accumulation of their future value Fv over n periods.

Future Value of an Annuity Due

Payments at the beginning of each period



Formula:

With an annuity due, payments are made at the beginning of each period. So the formula is slightly different. To calculate the future value of an annuity due, multiply the ordinary annuity formula by $(1 + i)$.

$$Fv = R \times \frac{(1+i)^n - 1}{i} \times (1 + i)$$

Where:

- R (PMT) = periodic payment
- i = interest rate per period
- n = number of payments

Example:

Suppose an individual decides to invest **125 000 DZD per year** over the next five years in a savings or investment account that offers an annual compound interest rate of **8%**. The contributions are made regularly at the end of each

year, and the invested amounts accumulate over time, generating interest that increases the future value of the annuity.

Solution:

In this example, the series of payments is a regular annuity in which the payments are made at the end of each period. The expected future value of this payment stream using the above formula is as follows:

$$Fv = 125\,000 \times \frac{(1+0.08)^5 - 1}{0.08} \times (1 + 0.08)$$

$$Fv = 792\,547.63 \text{ DZD}$$

II.2. Present Value of an Annuity Due

The present value of an annuity due uses the basic present value concept for annuities, except we should discount cash flow to time zero. The **present value** (PV) is also higher than that of an ordinary annuity because payments are received earlier. The formula for the present value of an annuity due is as follows:

$$PV = R * \left[\frac{1 - (1 + i)^{-n}}{r} \right] * (1 + i)$$

Where :

- **Pv**: Present Value
- **R (PMT)** : The payment amount per period
- **i**: The interest rate per period (divide the annual rate by the number of payments per year)
- **n**: The total number of compounding periods

Example:

An individual makes rental payments of 1200 DZD per month and wants to know the present value of their annual rentals over a 12-month period. The payments are made at the start of each month. The current interest rate is 8% per annum.

Solution:

Using the formula:

$$PV = R * \left[\frac{1 - (1 + i)^{-n}}{i} \right] * (1 + i)$$

$$PV = 1200 * \left[\frac{1 - (1 + 0.08/12)^{-12}}{0.08/12} \right] * (1 + 0.08/12)$$

$$PV \approx 13886.9044 \text{ DZD}$$

Comparison between an Ordinary Annuity and an Annuity Due

Criterion	Ordinary Annuity	Annuity Due
Definition	A series of equal payments made at the end of each payment period.	A series of equal payments made at the beginning of each payment period.
Timing of First Payment	At the end of the first period.	Immediately at the beginning of the first period.
Examples	Mortgage payments, most loan repayments, bond coupon payments.	Rent payments, lease payments, insurance premiums paid in advance.
Interest-Earning Period	Each payment earns interest for one period less.	Each payment earns interest for one additional period.
Future Value (FV)	Lower than the FV of an annuity due, all else being equal.	Higher than the FV of an ordinary annuity because each payment compounds for an extra period.
Present Value (PV)	Lower than the PV of an annuity due.	Higher than the PV of an ordinary annuity because payments are received earlier.
Future Value Formula	$FV = PMT * \frac{(1 + i)^n - 1}{i}$	$FV_{due} = PMT * \frac{(1 + i)^n - 1}{i} * (1 + i)$
Present Value Formula	$PV = PMT * \frac{1 - (1 + i)^{-n}}{i}$	$PV_{due} = PMT * \frac{1 - (1 + i)^{-n}}{i} * (1 + i)$
Relationship	Base case. $FV_{due} = FV_{ordinary}(1 + i)$ and $PV_{due} = PV_{ordinary}(1 + i)$	

Financial Tables and Their Use

In the field of **financial mathematics**, financial tables are an essential tool to simplify calculations related to the **time value of money**, whether for studying the **future value** or the **present value** of payments and installments. These tables typically include factors calculated based on the number of periods and the interest rate, such as:

- To determine the accumulation of capital with compound interest.

- To calculate the future value of equal annual payments (fixed annuities).
- To calculate the present value of regular payments.

Using these tables allows students or financial practitioners to:

1. **Determine interest rates or annual payments** quickly and accurately without performing lengthy calculations for each period.
2. **Make financial comparisons** between different financing options or between types of loans and deposits.
3. **Estimate approximate results with sufficient accuracy** when dealing with interest rates within a certain range, using **linear interpolation** between table values.

Thus, financial tables serve as a practical link between financial theory and computational practice, enabling efficient solutions for **installments, loans, and investment calculations**.

Example:

10 equal annuities of 1,000 DZD each have an accumulated value of 12,400 DZD. Knowing that the interest rate is between 4.5% and 4.75%, calculate the capitalization rate.

Preliminary Calculations:

$$V_n = a \frac{|(1+i)^n - 1|}{i} \Leftrightarrow 12400 = 1000 \frac{|(1+i)^{10} - 1|}{i} \Leftrightarrow \frac{12400}{1000} = 12,4 = \frac{|(1+i)^{10} - 1|}{i}$$

Reading the Financial table :

rate: 4,5%					
n	$(1+i)^n$	$(1+i)^{-n}$	$\frac{(1+i)^n - 1}{i}$	$\frac{1 - (1+i)^{-n}}{i}$	$\frac{i}{1 - (1+i)^{-n}}$
1	-----	-----	1,00000000	-----	-----
2	-----	-----	2,04500000	-----	-----
3	-----	-----	3,13702488	-----	-----
4	-----	-----	4,27819111	-----	-----
5	-----	-----	5,47070955	-----	-----
6	-----	-----	6,71689155	-----	-----
7	-----	-----	8,01915177	-----	-----
8	-----	-----	9,38001355	-----	-----
9	-----	-----	10,80211422	-----	-----
10	-----	-----	12,28820933	-----	-----
for n=10 and i=4,5% ; we read $\frac{(1+i)^n - 1}{i} = 12,28820933$					

Rate : 4,75%					
n	$(1+i)^n$	$(1+i)^{-n}$	$\frac{(1+i)^n - 1}{i}$	$\frac{1 - (1+i)^{-n}}{i}$	$\frac{i}{1 - (1+i)^{-n}}$
1	-----	-----	1,00000000	-----	-----
2	-----	-----	2,04750000	-----	-----
3	-----	-----	3,14475621	-----	-----
4	-----	-----	4,29413200	-----	-----
5	-----	-----	5,49810336	-----	-----
6	-----	-----	6,75926315	-----	-----
7	-----	-----	8,08032821	-----	-----
8	-----	-----	9,46414378	-----	-----
9	-----	-----	10,91369073	-----	-----
10	-----	-----	12,43209094	-----	-----
Pour n=10 et i=4,75% ; on lit $\frac{(1+i)^n - 1}{i} = 12,43209094$					

Source : Mohamed ZAATRI : Tables financières page 50 as cited in makhlouf ,2017

Reading the financial table :

$$i_1 = 0,045 \rightarrow y_1 = 12,28820933$$

$$i? = 0,045 \rightarrow y = 12,4$$

$$i_2 = 0,0475 \rightarrow y_2 = 12,43209094$$

Linear Interpolation:

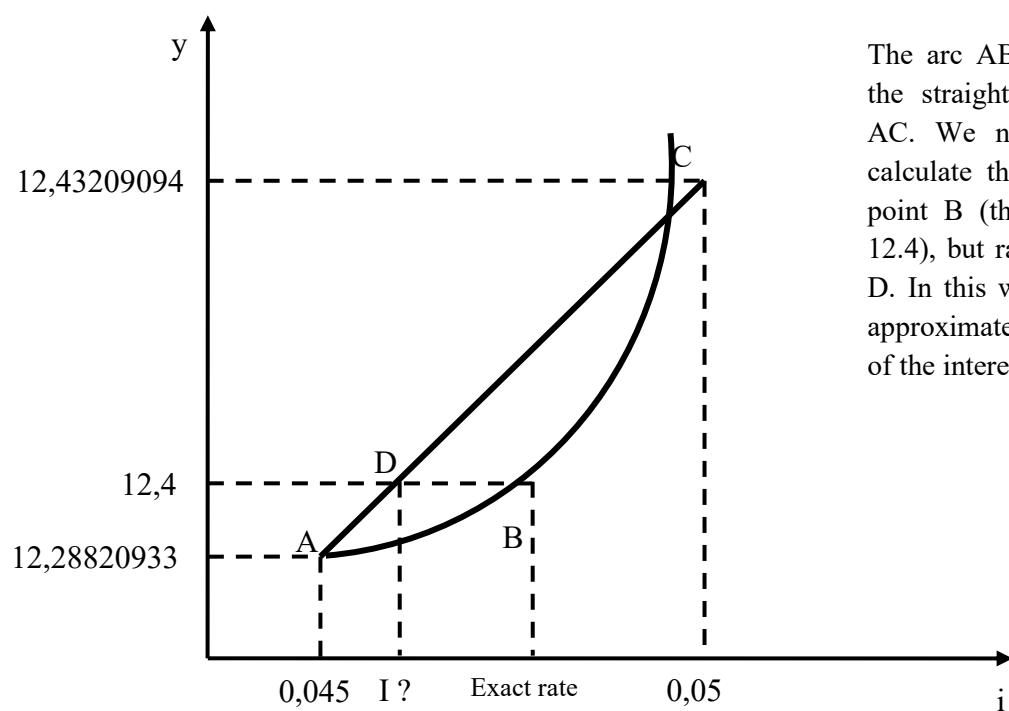
$$\frac{i - i_1}{i_2 - i_1} = \frac{y - y_1}{y_2 - y_1} \Leftrightarrow \frac{i - 0,045}{0,0475 - 0,045} = \frac{12,4 - 12,28820933}{12,43209094 - 12,28820933}$$

$$i = \frac{12,4 - 12,28820933}{12,43209094 - 12,28820933} (0,0475 - 0,045) + 0,045 = 0,0469$$

So $i = 4,96\%$

Notes :

1. The result is more accurate when the chosen rates (i_1 et i_2) are close.
2. Linear interpolation is used on: $\frac{(1+i)^n - 1}{i}$



The arc ABC is replaced by the straight line (or secant) AC. We no longer aim to calculate the x-coordinate of point B (the y-coordinate is 12.4), but rather that of point D. In this way, we obtain an approximate **underestimate** of the interest rate i

Tutorial Sheet n° 06: annuities

Exercise 1:

Multiple Choice (Circle the correct answer)

1. An ordinary annuity involves payments made:
A) At the beginning of each period
B) At irregular intervals
C) At the end of each period
D) Only once a year
2. If an ordinary annuity pays 500 MU monthly for 3 years at an interest rate of 6% compounded monthly, the number of periods is:
A) 3
B) 12
C) 36
D) 18

Exercise 2:

1. You deposit 200 DZD at the end of every month for 5 years in an account that earns 6% annual interest compounded monthly.

Calculate the future value.

2. You want to receive 1,000 DZD at the end of each year for 8 years. The discount rate is 5% per year.

Find the present value of the annuity.

3. You want to accumulate 10,000 DZD in 4 years with an account that pays 4% compounded annually.

How much should you deposit at the end of each year?

4. You pay 250 AD at the end of each quarter into an account that earns 3% compounded quarterly.

After how many quarters will the account balance reach 3,000 DZD?

Exercise 3: A debt of 1,000 DZD is contracted for one year at an annual interest rate of 10%, compounded monthly.

The debt will be repaid through 12 equal monthly payments made at the end of each month.

Calculate the amount of the monthly payment required to amortize the debt.

Exercise 4: The nominal interest rate is **9% with monthly compounding**. How many **end-of-month payments of 200 DZD** must be made to accumulate **5,000 DZD**?

Exercise 5

A person made 18 payments at the beginning of each year. The amount of the first six payments is 6,000 DZD, the amount of the next six payments is 2,000 DZD, and the amount of the last six payments is 10,000 DZD, at an interest rate of 2%.

– Calculate the accumulated value (future value).

If the interest rate changes and becomes 2% for the first six payments, 2% for the next six payments, and 2% for the last six payments,

– Calculate the accumulated balance

Solutions

Exercise 1 – Multiple Choice

C) At the end of each period

C) 36 (3 years $\times$ 12 months = 36 periods)

Exercise 2: – Computation Questions

1- Future Value of an Ordinary Annuity

Given:

PMT = 200 DZD

$i = 6\%$ annual $\rightarrow 0.06 / 12 = 0.005$ monthly

$n = 5$ years $\times 12 = 60$ months

Formula:

$$FV = R \left[\frac{(1 + i)^n - 1}{i} \right]$$

$$FV = 200 \left[\frac{(1 + 0.005)^{60} - 1}{0.005} \right]$$

$$FV \approx 200 \times 69.77 = \mathbf{13,954.69 \text{ DZD}}$$

2- Present Value of an Ordinary Annuity

Given: $R = \text{PMT} = 1,000 \text{ DZD}$ $i = 0.05$, $n = 8$

Formula :

$$Pv = R \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

$$Pv = 1000 \left[\frac{(1 + 0.05)^8 - 1}{0.05} \right]$$

$$PV \approx 1,000 \times 6.4631 = \mathbf{9,549.11 \text{ DZD}}$$

3- Find the Payment R (PMT)

1. **Given:**

$FV = 10,000 \text{ DZD}$

$i = 4\% = 0.04$ annually

$n = 4$

$$Fv = R \left[\frac{(1 + i)^n - 1}{i} \right]$$

$$10.000 = R \left[\frac{(1.04)^4 - 1}{0.04} \right]$$

$$R = \text{PMT} = 10,000 / 4.2465 \approx \mathbf{2,354.94 \text{ DZD}}$$

4- Determine the Number of Periods (n)

• **Given:**

$$R = \text{PMT} = 250 \text{ DZD}$$

$$\text{FV} = 3,000 \text{ DZD}$$

$$i = 3\% \text{ annually} \rightarrow 0.03 / 4 = 0.0075 \text{ quarterly}$$

Formula:

$$\begin{aligned} Fv &= R \left[\frac{(1 + i)^n - 1}{i} \right] \\ 3000 &= 250 \left[\frac{(1.0075)^n - 1}{0.0075} \right] \\ \frac{3000}{250} &= 12 = \left[\frac{(1.0075)^n - 1}{0.0075} \right] \end{aligned}$$

$$12 \times 0.0075 = (1.0075)^n - 1$$

$$0.09 = (1.0075)^n - 1$$

$$(1.0075)^n = 1.09$$

Take log:

$$n \times \log(1.0075) = \log(1.09)$$

$$n = \log(1.09) / \log(1.0075) \approx 0.0374 / 0.00323 \approx \mathbf{11.58 \approx 12 \text{ quarters}}$$

Therefore the number of periods is 12 quarters

Exercise 3

given:

- Loan: **1000 DZD**
- Annual interest rate: **10%**, compounded monthly
- Monthly rate is : $10\% / 12 = 0.008333333$
- Number of payments: $n=12$
- Payments at the end of each month (ordinary annuity)

Formula used

$$PV_n = R \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

$$R = PV_n \times i / [1 - (1 + i)^{-n}]$$

$$R = (1000 \times 0.008333333) / [1 - (1 + 0.008333333)^{-12}]$$

$$R = 87,5 \text{ DZD}$$

Monthly payment $\approx$ **87,92 DZD**

Exercise 4:**Given:**

- Payment $R=200$
- Future Value $FV=5000$
- Nominal interest rate = 9% compounded monthly
- Monthly rate:

$$i = 0.09/12 = 0.0075$$

$$FV = R \left[\frac{(1 + i)^n - 1}{i} \right]$$

$$5000 = 200 \left[\frac{(1 + 0.0075)^n - 1}{0.0075} \right]$$

$$n \approx 22.997$$

$$n \approx \mathbf{23 \text{ months}}$$

Chapter 06

Loans Amortization

Loans

In finance, there are two main types of funding sources.

The first is **bank financing (indivisible loans – *emprunt indivis*)**, where a single bank provides the loan, usually repaid in installments or at maturity with interest.

The second is **market financing (bond loans – *emprunt obligataire*)**, where funds are raised by issuing bonds purchased by many investors and traded in financial markets.

Thus, financing can be either direct from a bank or collective through financial markets.

Amortization of Loans

Amortization is the gradual repayment of a loan through regular payments that include both interest and principal. A common example is a mortgage, where the borrower repays the loan over time using the property as security. Each payment is split between interest and principal, and this breakdown is important for calculating the remaining balance and for tax purposes, as interest may be deductible.

Key Terms and Symbols in Loan Amortization

- **PV (Present Value)** refers to the initial amount borrowed from the lender.
- **n** represents the total number of payment periods, such as months or years.
- **i** is the interest rate per period. It can be annual, monthly, or according to any agreed schedule.
- **PMT** is the periodic payment. Depending on the loan type, it may remain constant (as in constant annuity loans) or vary (as in constant principal or deferred loans).

Interest (I_t) is the portion of the payment that covers the interest on the outstanding balance. For most amortized loans, it is calculated as the beginning balance multiplied by the interest rate:

$$I_n = B_{n-1} \times i$$

- **Principal payment (P_n)** is the portion of the payment that reduces the loan principal. For amortized loans, it is the payment minus interest:

$$P_n = PMT - I_t$$

In constant principal loans, this portion is fixed.

- **Beginning balance (B_{n-1})** is the outstanding loan amount at the start of the period, before the payment is made.

Ending balance (B_n) is the remaining loan principal after the payment has been applied. It is calculated as the beginning balance minus the principal payment:

$$B_n = B_{n-1} - P_n$$

- **Total payments** is the sum of all periodic payments over the loan term, and it may be constant or vary depending on the repayment method.

- **Total interest** is the sum of all interest payments made during the loan term. It is generally equal to the total payments minus the principal amount borrowed.
- An **amortization schedule** is a complete plan showing, for each period, the beginning balance, interest, principal, total payment, and ending balance, illustrating how the loan is gradually repaid

Loan Repayment Structures: Terms, Payment Patterns and Examples

N°	French Term	English Equivalent	Description	Payment (PMT)	Principal Repayment	Interest Payment	Remarks / Examples
1	Remboursement in fine	Lump-sum repayment / Bullet loan	The borrower pays interest only during the loan term and repays the entire principal at maturity .	Constant (interest only)	None during the term; full at the end	Constant	Typical of bonds or bullet loans.
2	Remboursement différé	Deferred repayment / Grace period loan	No payments during a grace period; repayment starts later.	No payments initially	Deferred	Interest accrued or capitalized	Used for student or development loans.
3	Remboursement par amortissements variables	Variable principal repayment	The principal portion varies according to agreed terms (e.g., linked to income, sales, etc.).	Variable	Variable	Variable	Customized loans.
4	Remboursement anticipé	Early repayment	The borrower repays part or all of the loan before maturity.	May include penalty	Varies	Varies	Common in flexible mortgage contracts.
5	Remboursement par amortissements constants	Repayment by constant principal repayments	The principal repaid each period is constant , while interest decreases over time.	Decreasing	Constant	Decreasing	Often used in corporate and long-term loans.
6	Remboursement par annuités constantes	Loan with constant annuities (equal payments)	The total payment (PMT) remains constant; the interest part decreases and the principal part increases progressively.	Constant	Increasing	Decreasing	Most common method (mortgage, car loan, etc.).

Three main repayment types- because those are the ones most commonly used and most pedagogically important:

Constant annuities (equal payments)

Constant principal repayments

Lump-sum (in fine / bullet) repayment

I- Loan with Constant Annuities (Equal Payments)

Formulas

$$PMT = PV \left[\frac{i}{1 - (1 + i)^{-n}} \right]$$

$$I_n = B_{n-1} \times i$$

$$P_n = PMT - I_t$$

$$B_n = B_{n-1} - P_n$$

Loan amortization table

Period (t)	Beginning Balance B_{n-1}	Interest I_n	Principal P_n	Total Payment PMT_n	Ending Balance B_n
1	$B_0 = PV$	$I_1 = B_0 \cdot i$	P_1	$PMT_1 = I_1 + P_1$	$B_1 = B_0 - P_1$
2	$B_1 = B_0 - P_1$	$I_2 = B_1 \cdot i$	P_2	$PMT_2 = I_2 + P_2$	$B_2 = B_1 - P_2$
...	...				
....					
.....					
S	$B_{S-1} = B_{S-2} - P_{S-1}$	$I_S = B_{S-1} \cdot i$	P_S	$PMT_S = I_S + P_S$	$B_S = B_{S-1} - P_S$
.....					
n-1	$B_{n-2} = B_{n-3} - P_{n-2}$	$I_{n-1} = B_{n-2} \cdot i$	P_{n-1}	$PMT_{n-1} = I_{n-1} + P_{n-1}$	$B_{n-1} = B_{n-2} - P_{n-1}$
n	$B_{n-1} = B_{n-2} - P_{n-1}$	$I_n = B_{n-1} \cdot i$	P_n	$PMT_n = I_n + m_n$	$B_n = B_{n-1} - P_n$

Example :

A producer plans to take out a 30,000 DZD loan for a period of 5 years, to be repaid annually at an annual interest rate of **5%**. The loan is to be repaid by constant Annuities.(constant periodic payment)

Required Work

1. Calculate the annual principal repayment (PMT).

2. Prepare the loan amortization schedule.

Solution

In **Year 1, Period 1**, the **beginning of period balance** is the full loan amount, which is **30,000 DZD** because no payments have been made yet.

Explanation:

1. The **beginning of period balance** represents the outstanding loan amount at the start of a period.
2. For Year 1, since it's the first period, the starting balance equals the loan principal.

From the amortization schedule, the details for Year 1 are:

- Beginning of Period Balance: 30,000 DZD
- Interest for the Period: $30,000 \times 5\% = 1,500 \text{ DZD}$

$$PMT = PV \left[\frac{i}{1 - (1 + i)^{-n}} \right]$$

$$PMT = 30\,000 \left[\frac{0.05}{1 - (1 + 0.05)^{-5}} \right]$$

- (PMT): **6,929.24 DZD**
- Principal Repayment: $6,929.24 - 1,500 = 5,429.24 \text{ DZD}$
- End of Period Balance: $30,000 - 5,429.24 = 24,570.76 \text{ DZD}$

Loan Amortization Schedule (Constant Annuities)

Period (Year)	Beginning Balance B_{n-1}	PMT Total Payment PMT	Interest I_n	Principal Repayment P_n	Ending Balance B_n
1	30,000.00	6,929.24	1,500.00	5,429.24	24,570.76
2	24,570.76	6,929.24	1,228.54	5,700.71	18,870.05
3	18,870.05	6,929.24	43.50	5,985.74	12,884.31
4	12,884.31	6,929.24	644.22	6,285.03	6,599.28
5	6,599.28	6,929.24	329.96	6,599.28	0.00

II- Loan with constant Principal repayment

A loan with constant principal repayments means the amount of principal paid each period is the same, while the total payment decreases over time

because the interest portion declines. For example, a **10,000 DZD loan** with **10 annual payments** and a **10% interest rate** would have a constant principal payment of **1,000 DZD each year** (10,000 DZD / 10). The first year's total payment would be **2,000 DZD** (1,000 DZD principal + 1,000 DZD interest on the initial \$10,000 balance), and the last year's total payment would be **1,035 DZD** (1,000 DZD principal + 35 DZD interest on the final 350 DZD balance)

Key takeaways

- **Constant Principal:** The principal portion of each payment is the same. It's calculated by dividing the total loan amount by the number of payments.
- **Declining Interest:** The interest portion decreases with each payment because it is calculated on a smaller and smaller outstanding balance.
- **Declining Total Payment:** Since the principal is constant and the interest declines, the total payment (principal + interest) gets smaller each period.
- **Less Total Interest:** This repayment method results in less total interest paid over the life of the loan compared to an equal total payment loan because the principal is paid down faster.
- **Higher Early Payments:** While the total interest is lower, this method requires higher payments in the early years of the loan.

Formulas

$$A = PV / n$$

$$I_n = B_{n-1} \times i$$

$$PMT_n = A - I_n$$

$$B_n = B_{n-1} - A$$

Where:

- **A:** fixed principal repaid each year
- **PMT_n** total payment, which decreases over time

Example**Loan Amount:** 10,000 DZD**Interest Rate:** 10% annual**Loan Term:** 10 years

Calculate the constant principal portion and prepare the amortization table .

Solution:

$$A = PV / n$$

$$A = 10.000/10 = 1000$$

principal repayment each year = 1,000 DZD

$$I_1 = B_0 \times i$$

$$10\,000 \times 0.1 = 1000 \text{ DZD}$$

$$I_2 = B_1 \times i$$

$$I_2 = 9\,000 \times 0.1 = 900 \text{ DZD}$$

$$PMT_n = A + I_n$$

$$PMT_1 = 1000 + 1000$$

$$\mathbf{PMT_1 = 2000 \text{ DZD}}$$

$$B_n = B_{n-1} - A$$

$$B_1 = B_0 - A$$

$$B_1 = 10\,000 - 1000$$

$$\mathbf{B_1 = 9000 \text{ DZD}}$$

Amortization schedule (constant Principal repayment)

Year	BeginningBalance	Interest	Principal (Fixed)	(PMT)	Ending Balance
1	10,000.00	1,000.00	1,000.00	2,000.00	9,000.00
2	9,000.00	900.00	1,000.00	1,900.00	8,000.00
3	8,000.00	800.00	1,000.00	1,800.00	7,000.00
4	7,000.00	700.00	1,000.00	1,700.00	6,000.00
5	6,000.00	600.00	1,000.00	1,600.00	5,000.00
6	5,000.00	500.00	1,000.00	1,500.00	4,000.00
7	4,000.00	400.00	1,000.00	1,400.00	3,000.00
8	3,000.00	300.00	1,000.00	1,300.00	2,000.00
9	2,000.00	200.00	1,000.00	1,200.00	1,000.00
10	1,000.00	100.00	1,000.00	1,100.00	0.00

III- Loan with Lump-Sum (Bullet / In Fine) Repayment

A lump-sum loan is a type of loan where you receive the entire amount of money in one single payment. The borrower then repays this principal amount plus interest in installments over a set period. A "lump-sum loan" can also refer to making an extra, single payment towards an existing loan to pay it off faster, often a mortgage.

Key Points

A lump-sum payment is an amount paid all at once, as opposed to an amount that is paid in installments.

Formulas

$$I_n = pv \times i$$

$$PMT = I_n \text{ for } n = 1, 2, 3, \dots, n-1.$$

$$PMT_n = I_n + pv$$

$$B_n = pv \text{ until final period}$$

Example

$PV = 10,000 \text{ DZD}$, $i = 8\%$ $n = 3$

Amortization schedule (Lump-Sum –in fine)

Year	Beg. Balance	Interest (8%)	Principal	Payment	End Balance
1	10,000.00	800.00	0.00	800.00	10,000.00
2	10,000.00	800.00	0.00	800.00	10,000.00
3	10,000.00	800.00	10,000.00	10,800.00	0.00

Note: The principal is repaid in full at the end, **and** interest is paid annually

Summary Comparison

Type	Payment (PMT)	Principal	Interest	Balance Trend	Use
Constant Annuities	Constant	Increasing	Decreasing	Gradually to zero	Mortgages, consumer loans
Constant Principal	Decreasing	Constant	Decreasing	Linear to zero	Corporate loans
Lump-Sum (Bullet)	Constant interest, final large payment	None until end	Constant	Constant until final	Bonds, financial markets

IV- Loan with Variable Amortization and Variable Annuities

A loan with variable principal repayments and variable installments is a loan in which both the principal and total payment amounts vary over time according to the repayment schedule or changes in interest rates.

Key Points

Both principal repayments and installments vary over time.

Example:

A loan of **500,000 DZD** is amortized over **five years** using **variable annual repayments**, with an **annual interest rate of 8%**. The **amortization schedule** is presented as follows:

Amortization schedule (Variable Amortization and Annuities)

Period n	Beginning Balance B_{n-1}	Interest I_n	Principal Payment P_n	Total Payment PMT_n	Ending Balance B_n
1	$(B_0) 500000$	$(I_1) 40000$	$(P_1) 100000$	$(PMT_1) 140000$	400.000
2	$(B_1) 400000$	$(I_2) 32000$	$(P_2) 110000$	$(PMT_2) 142000$	290.000
3	$(B_2) 290000$	$(I_3) 23200$	$(P_3) 140000$	$(PMT_3) 163200$	150.000
4	$(B_3) 150000$	$(I_4) 12000$	$(P_4) 90000$	$(PMT_4) 102000$	60.000
5	$(B_4) 60000$	$(I_5) 4800$	$(P_5) 60000$	$(PMT_5) 64800$	0
		112000	500000	612000	

Determination of the different elements:**a) Calculation of interest**

$$I_1 = B_0 i = 500000 \times 0.08 = 40000 \text{ DA}$$

$$I_2 = B_1 i = 400000 \times 0.08 = 32000 \text{ DA}$$

$$I_3 = B_2 i = 290000 \times 0.08 = 23200 \text{ DA}$$

$$I_4 = B_3 i = 150000 \times 0.08 = 12000 \text{ DA}$$

$$I_5 = B_4 i = 60000 \times 0.08 = 4800 \text{ DA}$$

$$I_n = V_{n-1}i$$

NOTE :

$$1. I_1 = (P_1 + P_2 + P_3 + P_4 + P_5)i$$

$$I_2 = (P_2 + P_3 + P_4 + P_5)i$$

$$I_3 = (P_3 + P_4 + P_5)i$$

$$I_4 = (P_4 + P_5)i$$

$$I_5 = P_5i$$

2. $I_n = P_n i$: as a reminder, this formula only works with the last repayment, because it is equal to the outstanding principal.

a) PMT calculation :

$$PMT_1 = P_1 + I_1 = P_1 + B_0 i = 100000 + 40000 = 140000 \text{ DA}$$

$$PMT_2 = P_2 + I_2 = P_2 + B_1 i = 110000 + 32000 = 142000 \text{ DA}$$

$$PMT_3 = P_3 + I_3 = P_3 + B_2 i = 140000 + 23200 = 163200 \text{ DA}$$

$$PMT_4 = P_4 + I_4 = P_4 + B_3 i = 90000 + 12000 = 102000 \text{ DA}$$

$$PMT_5 = P_5 + I_5 = P_5 + B_4 i = 60000 + 4800 = 64800 \text{ DA}$$

$$PMT_n = P_n + I_n = P_n + V_{n-1}i$$

b) Amortization calculation :

$$PMT_1 = P_1 + I_1 = P_1 + B_0 i \text{ Therefore, } M_1 = PMT_1 - B_0 i \text{ Or } M_1 = B_0 - B_1$$

$$PMT_2 = P_2 + I_2 = P_2 + V_1 i \text{ Therefore, } M_2 = PMT_2 - V_1 i \text{ ou } M_2 = B_1 - B_2$$

$$\text{And so on: } P_n = PMT_n - B_{n-1}i \text{ or } P_n = B_{n-1} - B_n$$

Application :

$$P_1 = 140000 - 500000 \times 0.08 = 100000 \text{ DA}$$

$$\text{Or } P_1 = 500000 - 400000 = 100000 \text{ DA}$$

c) Calcul de l'emprunt :

$$\text{PMT}_1 = V_0 i + P_1 \text{ donc } P_1 = \text{PMT}_1 - V_0 i \quad (1)$$

We have seen that $P_1 = B_0 - B_1$ Therefore: $B_1 = B_0 - P_1 \quad (2)$

We replace P_1 with its equivalent (1) :

$$B_1 = B_0 - P_1 \text{ Therefore: } V_1 = V_0 - (\text{PMT}_1 - V_0 i) = V_0(1+i) - \text{PMT}_1$$

$$\text{PMT}_2 + V_1 i + P_2 \text{ therefore: } (2) : P_2 + \text{PMT}_2 - B_1 i \quad (3)$$

We have seen that $P_2 = B_1 - B_2$ Therefore : $B_2 = B_1 - P_2 \quad (4)$

We replace P_2 with its equivalent (3) :

$$B_2 = B_1 - (\text{PMT}_2 - B_1 i) = B_1(1+i) - \text{PMT}_2$$

.....

$$B_n = B_{n-1}(1+i) - \text{PMT}_n$$

We want to express V_n as a function of V_0

$$B_1 = B_0(1+i) - \text{PMT}_1$$

$$B_2 = B_1(1+i) - \text{PMT}_2 = [B_0(1+i) - \text{PMT}_1](1+i) - \text{PMT}_2 = B_0(1+i)^2 - \text{PMT}_1(1+i) - \text{PMT}_2$$

$$B_3 = B_2(1+i) - \text{PMT}_3 = [B_0(1+i)^2 - \text{PMT}_1(1+i) - \text{PMT}_2](1+i) - \text{PMT}_3 = V_0(1+i)^3 - \text{PMT}_1(1+i)^2 - \text{PMT}_2(1+i) - \text{PMT}_3$$

.....

$$B_n = B_0(1+i)^n - \text{PMT}_1(1+i)^{n-1} - \text{PMT}_2(1+i)^{n-2} - \dots - \text{PMT}_n$$

At time n , the loan is fully repaid ($B_n = 0$) :

$$0 = B_0(1+i)^n - \text{PMT}_1(1+i)^{n-1} - \text{PMT}_2(1+i)^{n-2} - \dots - \text{PMT}_n$$

$$B_0(1+i)^n = \text{PMT}_1(1+i)^{n-1} + \text{PMT}_2(1+i)^{n-2} + \dots + \text{PMT}_n$$

$$B_0 = \frac{\text{PMT}_1(1+i)^{n-1} + \text{PMT}_2(1+i)^{n-2} + \dots + \text{PMT}_n}{(1+i)^n}$$

We factorize the numerator in order to eliminate the denominator:

$$B_0 = (1+i)^n \left[\frac{\text{PMT}_1(1+i)^{-1} + \text{PMT}_2(1+i)^{-2} + \dots + \text{PMT}_n(1+i)^{-n}}{(1+i)^n} \right]$$

$$B_0 = \text{PMT}_1(1+i)^{-1} + \text{PMT}_2(1+i)^{-2} + \dots + \text{PMT}_n(1+i)^{-n}$$

Application :

$$B_0 = 140000(1,08)^{-1} + 142000(1,08)^{-2} + 163200(1,08)^{-3} + 102000(1,08)^{-4} + 64800(1,08)^{-5} = 500000 \text{ DA}$$

Note:

Relation 3: The borrowed capital is equal to the sum of the present values of the annuities.

d) Calculation of the repaid principal (debt settled)

$$R_1 = P_1 = 100000 \text{ DA}$$

$$R_2 = P_1 + P_2 = 100000 + 110000 = 210000 \text{ DA}$$

$$R_3 = P_1 + P_2 + P_3 = 100000 + 110000 + 140000 = 350000 \text{ DA}$$

$$R_4 = P_1 + P_2 + P_3 + P_4 = 100000 + 110000 + 140000 + 90000 = 440000 \text{ DA}$$

$$R_5 = P_1 + P_2 + P_3 + P_4 + P_5 = 100000 + 110000 + 140000 + 90000 + 60000 = 500000 \text{ DA} = B_0$$

$$R_{k=1} = P_{k=1}$$

$$R_{k+1} = P_k + P_{k+1}$$

$$R_n = P_k + P_{k+1} + \dots + P_2 + \dots + P_n$$

d) Calculation of the outstanding principal (debt to be settled)

$$B_0 = P_1 + P_2 + P_3 + P_4 + P_5$$

$$= 100000 + 110000 + 140000 + 90000 + 60000 = 500000 \text{ DA}$$

Tutorial Sheet n° 07: Loan Amortization

Exercise 1:

From the amortization schedule of a conventional loan repaid through 8 equal annual instalments, we have extracted the following information:

- Interest for the second year = 18,251.1 DZD
- Interest for the third year = 16,327.3 DZD
- Outstanding balance at the end of the first year = 182,511 DZD

Required:

Calculate :

- The interest rate of the loan
- The first amortization (principal repaid in year 1)
- The fixed annual instalment
- The original amount of the loan
- The last row of the amortization schedule

Exercise 2

From the amortization schedule of a loan repaid over 6 equal annual instalments, the following information is available:

- Interest in the first year: 24,000 DZD
- Interest in the second year: 20,400 DZD
- Outstanding balance at the end of the first year: 480,000 DZD

Required:

1. Calculate the annual **interest rate**
2. Determine the **fixed annual instalment**
3. Find the **original amount of the loan**
4. Calculate the **amortization (principal repaid)** in the first year
5. Complete the **second row** of the amortization schedule

Solutions

Exercise 1:

Given :

- Interest in Year 2: $I_2=18,251.1$ DZD
- Interest in Year 3: $I_3=16,327.3$ DZD
- Outstanding balance at the end of Year 1: $B_1=182,511$ DZD
- Number of instalments: $n=8$

We are asked to find:

1. Annual interest rate i
2. First amortization A_1
3. Fixed annual instalment R
4. Original loan amount L
5. Last row of amortization schedule

Find the interest rate

Interest in any year is:

$$I_n = B_{n-1}xi$$

Where B_{n-1} is the outstanding balance at the **beginning of the year**.

We know:

$$I_2 = B_1xi$$

$$i = \frac{18251,1}{182511} = 0,1 = 10\%$$

- **Interest rate:** 10% per year
- **Original loan (L):** 200,000 DZD
- **Annual instalment (R):** 37,489 DZD
- **First principal repayment (A_1):** 17,489 DZD
- **Outstanding balances:** decrease each year from 182,511 DZD to 0 over 8 years
- **Amortization schedule:** fixed annual payments, interest portion decreases, principal portion increases

Exercise n°2:

Given:

- Annual interest rate: **4.25%**

- Original loan (L): **564,706 DZD**
- Fixed annual instalment (R): **108,706 DZD**
- First principal repayment (A_1): **84,706 DZD**
- Second year (row) of amortization schedule:
- Opening balance: 480,000 DZD
- Interest: 20,400 DZD
- Principal repaid: 88,306 DZD
- Instalment: 108,706 DZD
- Closing balance: 391,694 DZD

Year	Opening Balance (DZD)	Interest (DZD)	Principal Repaid (DZD)	Annual Instalment (DZD)	Closing Balance
1	564,706	24,000	84,706	108,706	480,000
2	480,000	20,400	88,306	108,706	391,694
3	391,694	16,678	92,028	108,706	299,666
4	299,666	12,740	95,966	108,706	203,700
5	203,700	8,653	100,053	108,706	103,647
6	103,647	4,402	104,304	108,706	0

Chapter 07: *Investment Project Selection*

Investment valuation basics

Investment Project Selection

The objective of studying **Investment Project Selection (or Capital Budgeting)** in financial mathematics is to evaluate alternative investment opportunities and make informed decisions regarding the acceptance or rejection of investment projects based on the **time value of money**.

More specifically, investment project evaluation seeks to answer the following questions:

- Is the project financially profitable?
- Which project should be selected among several investment alternatives?
- Do the expected returns adequately compensate for the invested capital and the associated risks?
- How long will it take to recover the initial investment?

Answering these questions enables investors and managers to allocate resources efficiently and select projects that maximize value.

Key points:

Investment project selection determines if an investment is profitable, compares alternatives, and assesses risk versus return.

1. Investment: Financial Perspective

Investment is a financial decision that involves allocating available financial resources today to acquire assets or undertake productive projects, with the aim of generating future financial benefits or cash inflows.

At its core, investment reflects a trade-off between present, certain liquidity and expected future returns, which are generally uncertain. This makes investment inherently linked to both time and risk.

In practice, investment may take the form of acquiring productive equipment, establishing industrial facilities, or purchasing intangible assets such as patents. All of these aim to enhance productive capacity and support economic growth.

Investment is also viewed as a long-term commitment that requires stable and permanent financing, usually through equity or medium- and long-term

sources of debt financing. In return, it must generate sufficient returns to cover its costs, recover the initial capital invested, and provide an adequate reward to providers of capital.

2. Main Characteristics of Investment

- **Return generation:** it must produce financial benefits that cover costs and generate additional profit.
- **Time dimension:** it extends over a long period, with outcomes distributed across several years.
- **Risk exposure:** future cash flows are uncertain.
- **Economic impact:** it contributes to increasing production capacity and creating value within firms and the broader economy.

Investment appraisal is a fundamental stage in the investment decision-making process. The literature in financial management generally addresses this topic through two main approaches: investment appraisal under conditions of certainty, and investment appraisal under conditions of uncertainty and risk. Each approach is based on a set of assumptions and analytical tools that correspond to the nature of the information available.

This chapter focuses on investment appraisal methods under conditions of certainty, where it is assumed that all relevant information relating to the investment project is known with precision. This includes expected future cash flows, the discount rate, and the economic life of the project, allowing for a deterministic evaluation in which uncertainty and risk are assumed to be absent.

3. Key Components of Investment Appraisal

Investment appraisal is based on a set of fundamental financial components that define the structure of any project evaluation. These elements are used to estimate cash flows, assess profitability, and support rational investment decision-making under the time value of money principle.

Initial Investment (I_0)

Includes the acquisition cost of the asset (net of VAT), additional acquisition costs such as transportation, installation, training, and customs duties, as well as the initial increase in working capital requirement (Working Capital / BFR).

Project Life (Economic Life)

Represents the period over which the investment is expected to be operated

and generate cash flows. In practice, depreciation life is often used as a proxy for economic life when estimating future cash flows.

Expected Cash Flows (Net Cash Flows)

Represent the net incremental cash inflows generated by the investment, calculated as operating revenues minus operating costs, excluding non-cash charges such as depreciation. These values are forecasted and subject to estimation assumptions.

Residual (Salvage) Value

Represents the estimated disposal value of the asset at the end of its economic life. It may be positive, zero, or negative depending on market conditions, asset condition, and disposal costs.

Discount Rate (Cost of Capital / Required Rate of Return)

Represents the opportunity cost of capital or the required rate of return demanded by investors. It is used to discount future cash flows to their present value in order to reflect the time value of money.

Tax Effects (Fiscal Impact)

Refer to the impact of corporate taxation on project cash flows, particularly the taxation of capital gains or tax savings from capital losses upon disposal, which affect the net terminal cash flow of the investment.

4. Projects selection methods

The main methods used under conditions of certainty include Net Present Value (NPV), Internal Rate of Return (IRR), Payback Period, and the Accounting Rate of Return (ARR). These techniques enable an objective comparison between investment alternatives and support the selection of the most economically viable project. The following table illustrates the main methods used in investment appraisal under conditions of certainty and highlights their respective objectives.

Order	English	French	Core Idea
1	Net Present Value (NPV)	Valeur Actuelle Nette (VAN)	Measures the difference between the present value of cash inflows and outflows
2	Internal Rate of Return (IRR)	Taux de Rentabilité Interne (TRI)	The discount rate that makes NPV = 0
3	Payback Period (PP)	Délai de récupération	The time required to recover the initial investment

4	Profitability Index (PI)	Indice de rentabilité	The ratio of the present value of cash inflows to the initial investment
5	Average Rate of Return (ARR)	Taux de rendement moyen	The average annual return compared to the investment cost

4.1 Net Present Value (NPV)

Let $CF_1, CF_2, \dots, CF_n$ denote the expected net cash flows generated by the project over time, and let I_0 represent the initial investment.

1. Naïve decision rule (not sufficient)

At first glance, a project may appear acceptable if the sum of all future inflows exceeds the initial cost:

$$-I + CF_1 + CF_2 + \dots + CF_n + R > 0.$$

Where R is the residual or terminal value of the project.

However, this criterion is **misleading**, because it ignores a fundamental financial principle:

A euro today is worth more than A euro in the future.

Therefore, simply adding cash flows across time without adjustment leads to an incorrect valuation.

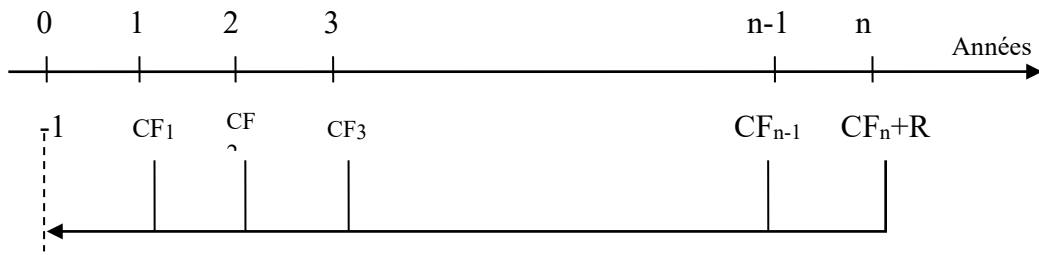
Correct methodology: Time Value of Money

To obtain a realistic evaluation, each cash flow must be adjusted (discounted) using an appropriate discount rate r , which reflects:

- the cost of capital,
- inflation expectations,
- project risk,
- and opportunity cost of funds.

Future cash flows are therefore converted into present value terms.

The financial flows related to the investment are presented chronologically as follows:



By applying a discount rate r to the various financial flows, the net present value (NPV) of the investment project is calculated.

$$NPV = -I + \frac{CF_1}{1+i} + \frac{CF_2}{(1+i)^2} + \frac{CF_3}{(1+i)^3} + \dots + \frac{CF_{n-1}}{(1+i)^{n-1}} + \frac{CF_n + R}{(1+i)^n}$$

where:

- CF_1 : net cash flow at time t
- r : discount rate
- R : terminal value (if any)
- I_0 : initial investment

Decision rule:

- $NPV > 0 \rightarrow$ accept the project (value creation)
- $NPV = 0 \rightarrow$ indifferent (break-even investment)
- $NPV < 0 \rightarrow$ reject the project (value destruction)

Thus, NPV directly measures the **increase in shareholder wealth**.

Example :

Given:

- Initial investment: $I_0 = 1000$ DZD
- Cash inflow after 1 year: $CF_1 = 1150$ DZD
- Discount rate: $r = 10\%$

$$NPV = -1000 + \frac{1150}{1 + 0.1} = 45.45 \text{ DZD} > 0.$$

Since $NPV > 0$, the project is acceptable because it generates a return above the required rate of 10%.

Special case: constant cash flows (annuity)

If cash flows are constant:

$$CF_1 = CF_2 = CF_3 = \dots \dots \dots CF_n = CF$$

Then NPV can be simplified using the present value of an annuity:

$$NPV = -I + CF \frac{1 - (1 + r)^{-n}}{i} + \frac{R}{(1 + r)^n}$$

Example :

An investment has an initial cost (**I**) of **100,000 DZD**, a zero residual value (**VR = 0**), and a discount rate (**t**) of **10%**. It generates positive annual cash flows of **25,000 DZD**, calculated as **30,000 DZD in inflows minus 5,000 DZD in outflows**. The investment becomes attractive in **period 6**, when the **Net Present Value (NPV)** turns positive.

Solution :

$$CF = 30.000 - 5.000 = 25.000 \text{ DZD}$$

$$NPV = -100\,000 + \frac{25000}{1 + 0.1} + \frac{25000}{1 + 0.1} + \frac{25000}{(1 + 0.1)^2} + \frac{25000}{(1 + 0.1)^3} + \dots$$

$$+ \frac{25000}{(1 + 0.1)^4} + \frac{25000}{(1 + 0.1)^5} + \frac{25000}{(1 + 0.1)^6}$$

$$NPV = 8.881$$

4.2. Payback Period (DR)

The payback period is a simple investment appraisal method that measures the **time required to recover the initial capital invested** through the cash inflows generated by the project. It determines the point in time at which the **cumulative cash flows equal the initial investment**.

Steps of the method

1. Determine the annual cash flows of the project.
2. Compute cumulative cash flows over time.
3. Identify the period in which the initial investment is fully recovered.

Decision rule

- If the payback period is **shorter than the maximum acceptable period set by the firm**, the project is accepted.
- If it is **longer**, the project is rejected.

Note :

This method does not measure the project's true profitability; it only focuses on **the speed of capital recovery**. It also ignores cash flows occurring after the payback period and does not take into account the time value of money in its basic version.

The payback method is a quick and simple tool used to assess liquidity and risk, but it is generally used alongside other investment appraisal methods such as **NPV and IRR**.

$$PP = \frac{I}{CF}$$

Where:

- **P:** Payback Period
- **I₀:** Initial investment (cash outflow at t=0)
- **CF:** Annual net cash flow (constant cash inflows)

Interpretation of the Payback Period

Independent projects

For independent projects, the decision rule is to **accept the project if:**

$$PP \leq PP^*$$

Where is the **maximum acceptable payback period (cut-off period)** defined by management.

Mutually exclusive projects

For mutually exclusive projects, the decision rule is:

- Choose the project with the **lowest PP**, such that:

$$PP \leq PP^*$$

Note :

PP *denotes the **maximum acceptable payback period** (or cut-off period) established by the firm as a benchmark for evaluating investment projects.

4.3. Internal Rate of Return (IRR)

The Internal Rate of Return (IRR) is the discount rate that makes the Net Present Value (NPV) of an investment project equal to zero. In other words, it is the rate at which the present value of future cash inflows equals the initial investment cost.

It is determined by solving the following equation:

$$NPV = 0 = -I + \sum_{1 \leq p \leq N} \frac{CF_p}{(1 + IRR)^p} \Rightarrow \sum_{1 \leq p \leq N} CF_p \times (1 + IRR)^{-p} = I$$

Decision Rule

The IRR is compared with the required rate of return or the firm's cost of capital r :

- If

$$IRR > r$$

the project should be **accepted**.

- If

$$IRR = r$$

the project is **indifferent (neutral)**, since its NPV equals zero.

- If

$$IRR < r$$

The project should be **rejected**.

Economic Interpretation

The IRR represents the annual rate of return generated by the investment project. The higher the IRR relative to the firm's cost of capital or required rate of return, the more attractive the investment becomes.

Key Points :

The Internal Rate of Return is one of the most widely used investment appraisal techniques because it identifies the maximum financing cost that a project can bear while remaining economically viable.

Tutorial Sheet n° 08: Investment project selection

Exercise 1:

A firm is evaluating **three investment projects (A, B, and C)**.

Each project has an **economic life of 6 years** and a **discount rate of 15% per year**.

The initial investment for each project is:

- Project A: **80,000 DZD**
- Project B: **85,000 DZD**
- Project C: **90,000 DZD**

The expected **net annual cash flows** are given in the table below

<i>Year</i>	<i>Project A</i>	<i>Project B</i>	<i>Project C</i>
1	20, 000	25, 000	18, 000
2	22, 000	24, 000	20, 000
3	24, 000	23, 000	22, 000
4	26, 000	22, 000	24, 000
5	28, 000	21, 000	26, 000
6	30, 000	20, 000	28, 000

- 1- Calculate the **Net Present Value (NPV)** of each project using a discount rate of **15%**.
- 2- Determine which project should be selected based on the NPV criterion.
- 3- Justify your decision economically.

Exercise 2:

A company is considering replacing an existing machine, which can be sold today for **1,800 DZD**, representing its residual (salvage) value. The new investment project is expected to generate an increasing stream of cash flows over **5 periods**. The cash flow in the first period is **600 DZD**, and it grows at a constant geometric rate of **80% per period**.

The average market interest rate (geometric mean rate) is **7% per year**.

- Calculate the **Net Present Value (NPV)** of the investment project.

Solutions

Exercise 1:

1. NPV formula

For a project with different annual cash flows:

$$NPV = -I + \frac{CF_1}{1+i} + \frac{CF_2}{(1+i)^2} + \frac{CF_3}{(1+i)^3} + \dots + \frac{CF_{n-1}}{(1+i)^{n-1}} + \frac{CF_n + R}{(1+i)^n}$$

where:

- I = initial investment
- CF_t = cash flow in year t
- $i=15\%$

Project A

Initial investment = **80,000. 00 DZD**

$$NPV = -80\,000 + \frac{20\,000}{1+0.15} + \frac{22\,000}{(1+0.15)^2} + \frac{24\,000}{(1+i)^3} + \frac{26\,000}{(1+i)^4} + \frac{28\,000}{(1+i)^5} + \frac{30\,000}{(1+i)^6}$$

Present value of cash flows:

<i>Year</i>	<i>Cash flow</i>	<i>Present value</i>
1	20,000	17,391.30
2	22,000	16,635.17
3	24,000	15,780.38
4	26,000	14,865.58
5	28,000	13,920.96
6	30,000	12,969.84
Total		91,563.23

Therefore: $NPV_A = 91,563.23 - 80,000.00 = 11,563.23 \text{ DZA}$

Project B

Initial investment = **85,000. 00 DZD**

$$NPV = -85\,000 + \frac{25\,000}{1 + 0.15} + \frac{24\,000}{(1 + 0.15)^2} + \frac{23\,000}{(1 + 0.15)^3} + \frac{22\,000}{(1 + 0.15)^4} + \frac{21\,000}{(1 + 0.15)^5} + \frac{20\,000}{(1 + 0.15)^6}$$

Present value of cash flows:

<i>year</i>	<i>Cash flow</i>	<i>Present value</i>
1	25,000	21,739.13
2	24,000	18,147.46
3	23,000	15,122.87
4	22,000	12,578.57
5	21,000	10,440.72
6	20,000	8,646.56
Total		86,675.31

Therefore: $NPV_B = 86,675.31 - 85,000.00 = 1,675.31$ DZA

Project C

Initial investment = **90,000.00 DZD**

$$NPV = -90\,000 + \frac{18\,000}{1 + 0.15} + \frac{20\,000}{(1 + 0.15)^2} + \frac{22\,000}{(1 + 0.15)^3} + \frac{24\,000}{(1 + 0.15)^4} + \frac{26\,000}{(1 + 0.15)^5} + \frac{28\,000}{(1 + 0.15)^6}$$

Present value of cash flows:

<i>year</i>	<i>Cash flow</i>	<i>Present value</i>
1	18,000	15,652.17
2	20,000	15,122.87
3	22,000	14,465.35
4	24,000	13,722.07
5	26,000	12,926.61
6	28,000	12,105.18
Total		83,994.25

Therefore: $NPV_C = 83,994.25 - 90,000.00 = -6,005.75$ DZA

<i>Project</i>	<i>Initial investment</i>	<i>PV of cash flows</i>	<i>NPV (DZD)</i>
A	80,000	91,563.23	+11,563.23
B	85,000	86,675.31	+1,675.31
C	90,000	83,994.25	-6,005.75

If the firm must select **one mutually exclusive project**, the NPV criterion gives:

Select Project A

because it has the **highest NPV (+11,563.23 DZD)**.

Economic justification: Project A provides the greatest increase in the firm's wealth after taking into account both the timing of the cash flows and the 15% opportunity cost of capital.

Exercise 2

1. Calculate the cash flows

$$CF_1 = 600$$

$$CF_2 = 600(1.80) = 1,080$$

$$CF_3 = 1,080(1.80) = 1,944$$

$$CF_4 = 1,944(1.80) = 3,499.20$$

$$CF_5 = 3,499.20(1.80) = 6,298.56$$

2. Calculate the present value of the cash flows

For a project with different annual cash flows:

$$NPV = -I + \frac{CF_1}{1+i} + \frac{CF_2}{(1+i)^2} + \frac{CF_3}{(1+i)^3} + \dots + \frac{CF_{n-1}}{(1+i)^{n-1}} + \frac{CF_n + R}{(1+i)^n}$$

Therefore:

$$NPV = -1800 + \frac{600}{1 + 0.07} + \frac{1,080}{(1 + 0.07)^2} + \frac{1,944}{(1 + 0.07)^3} + \frac{3,499.20}{(1 + 0.07)^4} + \frac{6,298.56}{(1 + 0.07)^5}$$

Period	Cash Flow	Present Value
1	600.00	560.75
2	1,080.00	943.31
3	1,944.00	1,586.88
4	3,499.20	2,669.52
5	6,298.56	4,490.79
Total		10,251.25

3. Calculate the NPV

The existing machine can be sold today for **1,800 DZD**. Therefore, replacing it involves an initial opportunity cost of 1,800 DZD:

$$NPV = -1,800 + 10,251.25$$

$$NPV = 8,451.25 \text{ DZD}$$

Therefore, **based on the NPV criterion, the replacement project is financially acceptable.**

Important: The exercise does **not** provide the purchase price of the new machine. Consequently, the only initial investment that can be incorporated into the NPV calculation from the information given is the **1,800 DZD opportunity cost represented by the forgone salvage value.**

Practical Exercises

Note on the Exercises

The following exercises are adapted from **(Leamy maths community, 2019)** and are based on real-life financial situations. They illustrate how financial mathematics is applied in practice, including lottery payments, pension schemes, loans, and investments. Some examples are based on specific financial contexts and practices that may differ from those used in Algeria.

Compound Interest

1. Find the future value of an investment of 5000€, earning interest at an AER (Annual Equivalent Rate) of 2.5% for 8 years.
2. 25000€ is invested at an AER of 3.2%. Find the future value after 15 years.
3. What is the monthly interest rate for an investment with an AER of 3.5%?
4. What is the monthly interest rate for an investment with an AER of 4.2%?
5. What is the AER of an investment earning 0.2% interest per month?
6. What is the quarterly interest rate for an investment with an AER of 5.3%?
7. What is the AER of an investment earning interest at 1.1% per quarter?
8. What is the future value of 10000€ invested for 27 months with an AER of 3.5%?
9. 3500€ was invested at an AER of 4.3%. What will the investment be worth after 5 years and 5 months?
10. 500€ was invested at an AER of 3.8%. After how long will the investment be valued at 800€?
11. How long will it take an investment to double if it is earning interest at 5.1% per annum?
12. An investment of 3500€ amounted to 4937.10€ after 10 years. What was the AER?
13. An investment of 5000€ amounted to 6895.66€ after 8 years. What was the AER?
14. Find the present value of 2,000€ to be paid in 3 years time, at an AER of 3.7%.
15. What amount of money needs to be invested at 2.8%, in order to have 10000€ in five years?

16. As part of a business agreement, Company A will pay Company B 250000€ in 10 years. What is the present value of this payment, if the AER is 4.2%?
17. What is the present value of 3600€ to be paid in 34 months time at rate of 5%?
18. As part of a corporate bond, Company X must pay its creditors 3000€ in 3 years and 3 months, and a further 5000€ in 5 years and 5 months. What is the total present value of these payments, assuming an AER of 4.4%?
19. Kevin will turn 21 in 2 years and 7 months. Upon his 21st birthday he is entitled to an inheritance of 21000€. What is the present value of this inheritance, assuming an AER of 3.8%?
20. Mary borrows 20000€ to buy a car. If she makes no repayments, how much will she owe the bank in 5 years time if the AER is 4.6%?
21. Jack borrows 15000€ from the bank for 5 years. The AER for the first 3 years is 4.5%. After 3 years he pays back 10000€, so the AER drops to 3.5%. How much will Jack need to fully repay the loan after the fifth year? How much interest did he pay?

Depreciation

1. A car purchased for 30000€ depreciates at 15% per annum. What will the car be worth five years after it is bought?
2. A lorry which cost 75000€ new, depreciates at 20% per annum for the first two years, then at 10% per annum after two years. What will the lorry be worth after 5 years?
3. A machine bought for 500000€ new was worth 220000€ after 7 years. What was the annual depreciation rate?
4. A lorry bought for 125000€ depreciates at a rate of 17% per annum. If the lorry is to be sold before the price drops below 75000€, how long before it is sold?
5. A company buys a factory machine for 250000€. The value of this machine will depreciate at 22% per annum. At the same time the company also invests 50000€ into a fund with a return of 5% per annum.
 - i. What will the factory machine be worth after 6 years?
 - ii. What will be the value of the investment after 6 years?
 - iii. If inflation is 1.8% per annum. How much will a new version of the same factory machine cost in 6 years?
 - iv. After 6 years, the company will sell the original machine, and upgrade by buying a new one. They will cash in their investment and use this money plus the money they receive from selling the old machine, to buy the new machine. How much extra funding will they need to buy the new machine?

Financial Series

a-Saving Schemes

1. Anna will deposit 1000€ at the beginning of every year for 10 years. Calculate the value of her savings at the end of 10 years, if the AER is 3%.

2. Calculate the future value of 48 monthly installments of 100€, at an AER of 5.1%.

What is the total interest earned on this investment?

3. 50€ is deposited weekly into an account paying 4.3% interest per annum. How much will the savings be worth after a year?

4. When Michael turned 18 he decided to open a bank account and deposit 50€ at the beginning of every month. If the AER is 4.2%, how much will Michael have in his account on his 21st birthday?

5. When Jean was born, her parents set up a college fund for her. They plan to put a set amount into the bank at the beginning of every month. They want to have \$250000 in Jeans college fund on her 18th birthday. How much should they deposit every month if the fixed rate on the fund is 6%?

6. A couple are saving for a deposit for a house. They want to deposit the same amount at the beginning of every month. They would like to have 20000€ saved after 3 years.

If the bank are offering a savers rate of 3.8%, how much do the couple need to deposit every month to reach their target?

7. A company sets up a sinking fund for their employees. How much has to be paid into the fund every week, if the fund is to be worth 50000€ in 5 years time? Assume an AER of 5%.

b-Bonds and Assets

8. A ten year bond offers a 30% total return on the investment. What is the AER of this bond?
9. A twenty year bond offers a 100% total return on the investment. What is the AER of this bond?
10. Company Y has issued ten year corporate bonds to raise money. They will have to pay their creditors 1000€ at the end of every year for the next ten years. What is the fair price for this bond, if the expected market interest rate is 6% per annum?
11. Company Z has issued ten year corporate bonds to raise money. They will have to pay their creditors 500€ at the end of every year for ten years, plus pay an additional lump sum of 10000€ at the end of the ten years. What is the fair price for this bond, if expected market interest rates are at 7%?
12. A technology company issues a five year corporate bond to raise money. Because technology companies are seen as high risk investments, they will have to pay a market interest rate of 12.5% per annum. As part of the structure of the bond, they will pay their creditors 5000€ per quarter for 5 years, plus a lump sum of 50000€ at the end of the five years. What is the fair market price for this bond?

c-Lottery Payments

13. In the USA lottery winners have a choice of taking a lump sum now or receiving regular payments for a certain length of time. Matt has just won the lottery and has the choice of receiving \$1m now, or \$6000 at the beginning of each month for the next 20 years. Assuming an AER of 5%, which option should Matt choose?
14. Emma just won the New York powerball lottery. She can choose between a lump sum of \$15m now, or receive payments of \$37000 at the

beginning of every week for the next 10 years. Which option should she choose if the interest rate is 4% per annum?

15. After winning the lottery in the USA, Amy is offered a choice between a lump sum of \$10m now or receive a set payment at the beginning of every month for 20 years. If the expected interest rate is 5%, what should the monthly payments be, so the sum of the present values of these payments is the same value as the lump sum?

d-Loans

16. Calculate the monthly loan repayments on a 35 year mortgage of 300000€, at 3.5% interest.

17. Calculate the monthly loan repayments on a 20 year business loan of 155000€, at an interest rate of 7.5%.

18. Jessica wants to take out a 35 year mortgage. She can afford to make repayments of 800€ per month. If the fixed interest rate is 4%, what is the largest mortgage she can take out?

19. A small business wants to take out a 10 year commercial loan to buy a new premises.

They can afford to make repayments of 2500€ a month. If the interest rate on small business loans is 8%, what is the largest loan that the business can afford?

e-Pension Schemes

20. Mark is thinking of setting up a retirement fund for his pension. The pension scheme is designed in such a way that he will receive annual payments of 30000€, for twenty five years, with the first payment on the day he retires. The fixed interest rate for the entire pension scheme is 5%.

(a) How much money needs to be in the retirement account on the day that Mark retires, in order to fund the twenty five annual payments of 30000€?

(b) Mark starts saving for his pension 20 years before he will retire. How much must he deposit annually, at the beginning of each year, to ensure he has adequate funds in his retirement account on the day that he retires?

21. Marie is setting up her retirement fund. She will retire in twenty five years. Her pension will provide monthly payments of 2500€, at the beginning of every month, beginning on the day she retires. Her pension payments will last for twenty years. The expected rate of interest is 6%.

(a) How much money needs to be in her retirement account on the day that Marie retires, in order to fund her pension payments?

(b) Marie will make equal deposits at the beginning of every week in order to save for her retirement. How much should she deposit every week, to ensure there is adequate funds in her retirement account on day that she retires?

22. How much must John save at the beginning of every month, for twenty years up until retirement, in order to fund a pension that will pay him 3000€ at the beginning of every month, for fifteen years, from the day he retires? Let the AER be a fixed 5.5%.

Exam Questions

Most lottery games in the USA allow winners of the jackpot prize to choose between two forms of the prize: an annual-payments option or a cash-value option. In the case of the New York Lotto, there are 26 annual payments in the annual payments option, with the first payment immediately, and the last payment in 25 years' time. The payments increase by 4% each year. The amount advertised as the jackpot prize is the total amount of these 26 payments. The cash value option pays a smaller amount than this

- (a) If the amount of the first annual payment is A , write down, in terms of A , the amount of the second, third, fourth and 26th payments.
- (b) The 26 payments form a geometric series. Use this fact to express the advertised jackpot prize in terms of A .
- (c) Find, correct to the nearest dollar, the value of A that corresponds to an advertised jackpot prize of \$21.5 million.
- (d) A winner who chooses the cash-value option receives, immediately, the total of the present values of the 26 payments. The interest rate used for the present-value calculations is 4.78%. We want to find the cash value of the prize referred to in part (c).

- i. Complete the table below to show the actual amount and the present value of each of the first three annual payments.

Payment Number	time to payment (years)	actual amount	present value
1	0		
2	1		
3	2		

- ii. Write down, in terms of n , an expression for the present value of the n th annual payment.

- iii. Find the amount of prize money payable under the cash-value option. That is, find the total of the present values of the 26 annual payments. Give your answer in millions, correct to one decimal place.
- (e) The jackpot described in parts (c) and (d) above was won by an Irish woman earlier this year. She chose the cash-value option. After tax, she received \$7.9 million. What percentage of tax was charged on her winnings?

SOLUTIONS

Compound Interest

1. 6092.01€
2. 40099.18€
3. 0.287 %
4. 0.343 %
5. 2.43 %
6. 1.3 %
7. 4.47 %
8. 10804.78€
9. 4396.51€
10. 12.6 years
11. 13.9 years
12. 3.5 %
13. 4.1 %
14. 1793.47€
15. 8710.33€
16. 165677.23€
17. 3135.21€
18. 6568.04€
19. 19071.12€
20. 25043.12€
21. 7624.44€; 2624.44€

Depreciation

1. 13311.16€
2. 34992€
3. 11 %

4. 2.7 years

5. i. 56299.90€

ii. 67004.78€

iii. 278244.56€

iv. 154939.88€

Financial Series

1. 11807.80€

2. 5321.85€; 521.85€

3. 2656.58€

4. 1919.08€

5. 653.05€

6. 524.24€

7. 169.72€

8. 2.66 %

9. 3.53 %

10. 7360.09€

11. 8595.28€

12. 102213.96€

13. P.V. of series = 921399.32€

Take the lump sum

14. P.V. of series = 15921433.44€

Take the series of payments

15. 65118.35€

16. 1230.34€

17. 1225.45€

18. 182442€

19. 208580.98€

20. (a) 443959.25€

(b) 12787.12€

21. (a) 355181.26€

(b) 120.84€

22. 863.61€

Exam Question: Solution(a) $A(1.04); A(1.04)^2; A(1.04)^3; A(1.04)^5$ (b) $44:31174462 \times A$

(c) \$ 485199

(d) i. The table showing the actual amount and the present value of each payment.

payment number	time to payment (years)	actual amount	present value
1	0	485,199	485,199
2	1	504,606.96	481,587.10
3	2	524,791.24	478,002.02

ii. $485199 \left(\frac{1.04}{1.0478} \right)^{n-1}$

(e) \$11.5m. Tax rate 31.4 %.

Conclusion

This booklet has been designed as both a theoretical and practical guide to understanding the fundamental concepts of financial mathematics. Through its structured lessons, applied examples, and varied exercises, it aims to help students grasp the key principles governing short- and long-term financial operations and to develop their ability to apply them in real-world situations.

Beyond mere calculation techniques, financial mathematics trains the mind to think logically, analyze financial decisions critically, and make informed choices - skills that are indispensable in today's economic and banking environment.

It is hoped that this work will serve as a valuable reference and a solid foundation for students pursuing further studies in finance, accounting, and management, while also encouraging continuous learning and deeper exploration of the fascinating relationship between time and money.

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Benjamin Franklin, *Advice to a Young Tradesman, Written by an Old One* (1748). The line is "Remember that time is money."